Grade 8 Chapter 7: Rational and Irrational Numbers

In this chapter we first review the real line model for numbers, as discussed in Chapter 2 of seventh grade, by recalling how the integers and then the rational numbers are associated to points in the line. Having associated a point on the real line to every rational number, we ask the question, do all points correspond to a rational number? Recall that a point on the line is identified with the length of the line segment from the origin to that point (which is negative if the point is to the left of the origin). Through constructions (given by “tilted” squares), we make an observation first made by the Pythagorean society 2500 years ago that there are lengths (such as the diagonal of a square with side length 1) that do not correspond to a rational number. The construction produces numbers whose squares are integers; leading us to introduce the symbol $\sqrt{A}$ to represent a number whose square is $A$. We also introduce the cube root $\sqrt[3]{V}$ to represent the side length of a cube whose volume is $V$. The technique of tilted squares provides an opportunity to observe the Pythagorean theorem: $a^2 + b^2 = c^2$, where $a$ and $b$ are the lengths of the legs of a right triangle, and $c$ is the length of the hypotenuse.

In the next section we return to the construction of a square of area 2, and show that its side length ($\sqrt{2}$) cannot be equal to a fraction, so its length is not a rational number. We call such a number an irrational number. The same argument works for $\sqrt{5}$ and other lengths constructed by tilted squares. It is a fact that if $N$ is an integer, either it is a perfect square (the square of an integer), or $\sqrt{N}$ is not a quotient of integers; that is $\sqrt{N}$ is an irrational number.

In the next section we turn to the question: can we represent lengths that are not quotients of integers, somehow by numbers? The ancient Greeks were not able to do this, due mostly to the lack of an appropriate system of expressing lengths by their numerical measure. For us today, this effective system is that of the decimal representation of numbers (reviewed in Chapter 1 of seventh grade).

We recall that by dividing an interval into ten equal parts, we move on to the next decimal point. To be precise: Given a point $x$ on the real line count the number of unit interval that can be put completely in the interval from the origin to $x$. This is the integral part of the number associated to $x$ (negative if $x$ is left of the origin). Now divide the unit interval into ten equal parts, and count how many fit in the space left over. That digit goes in the tenth place. repeating this for as many times as necessary, we can associate to every (terminating) decimal a point on the line.

A rational number is represented by a terminating decimal only if the denominator is a product of twos and fives. Thus many rational numbers (like $1/3, 1/7, 1/12,...$) are not represented by terminating decimals, but they are represented by repeating decimals, and similarly, repeating decimals represent rational numbers. We now view the decimal expansion of a number as providing an algorithm for getting as close as we please to its representing point on the line through repeated subdivisions by tenths. In fact, every decimal expansion represents a point on the line, and thus a number, and unless the decimal expansion is
terminating or repeating, it is irrational.

The question now becomes: can we represent all lengths by decimal expansions? We start
with square roots, and illustrate Newton’s method for approximating square roots: Start
with some reasonable estimate, and follow with the recursion
\[ a_{\text{new}} = \frac{1}{2} \left( a_{\text{old}} + \frac{N}{a_{\text{old}}} \right). \]

Through examples, we see that this method produces the decimal expansion of the square
root of \( N \) to any required degree of accuracy.

Finally, we point out that to do arithmetic operations with irrational as well as rational
numbers, we have to be careful: to get within a specified number of decimal points of
accuracy we may need much better accuracy for the original numbers.

Section 7.1. Representing numbers geometrically

Interpret numbers geometrically through the real line.

First, let us recall how to represent the rational number system by points on a line. With
a straight edge, draw a horizontal line. Given any two points \( a \) and \( b \) on the line, we say
that \( a < b \) if \( a \) is to the left of \( b \). The piece of the line between \( a \) and \( b \) is called the interval
between \( a \) and \( b \). It is important to notice that for two different points \( a \) and \( b \) we must have
either \( a < b \) or \( b < a \). Also, recall that if \( a < b \) we may also write this as \( b > a \).

Pick a point on a horizontal line, mark it and call it the origin, denoted by \( 0 \). Now place a
ruler with its left end at \( 0 \). Pick another point (this may be the 1 cm or 1 in point on the
ruler) to the right of \( 0 \) and denote it as \( 1 \). We also say that the length of the interval between
\( 0 \) and \( 1 \) is one \( \text{pr} \) one unit. Mark the same distance to the right of \( 1 \), and designate that
endpoint as \( 2 \). Continuing on in this way we can associate to each positive number a point
on the line. Now mark off a succession of equally spaced points on the line that lie to the
left of \( 0 \) and denote them consecutively as \(-1, -2, -3, \ldots\). In this way we can imagine
all integers placed on the line.

We can associate a half integer to the midpoint of any interval; so that the midpoint of the
interval between \( 3 \) and \( 4 \) is \( 3.5 \), and the midpoint of the interval between \(-7 \) and \(-6 \) is \(-6.5 \). If
we divide the unit interval into three parts, then the first part is a length corresponding to
\( 1/3 \), the first and second parts correspond to \( 2/3 \), and indeed, for any integer \( p \), by putting
\( p \) copies end to end on the real line (on the right of the origin is \( p > 0 \), and on the left if
\( p < 0 \), we get to the length representing \( p/3 \). We can replace \( 3 \) by any positive integer \( q \),
by constructing a length which is one \( q \)th of the unit interval. In this way we can identify
every rational number \( p/q \) with a point on the horizontal line, to the left of the origin if \( p/q \)
is negative, and to the right if positive.

Now draw a vertical line through the origin, and do the same on that line, using the same
interval as the unit interval. Now, to every pair of rational numbers \((a, b)\) we can associate a
point in the plane: Go along the horizontal (the $x$-axis) to the point $a$. Now, set the pins of a protractor at the endpoints of the interval from the origin to the point $b$, and measure this out on the vertical line through $a$, arriving at the point to be identified with $(a, b)$. Finally, we can measure the length of any line segment on the plane using a protractor, as follows: set the pins of a protractor at the endpoints of the line segment and then measure that length along the horizontal (or vertical) axis, with one pin at the origin. The other pin lands on or near some point representing a fraction $p/q$. The closest such fraction is an estimate for the length of the line segment.

**Example 1.** In figure 1 the unit lengths are half an inch each. What are the lengths of the sides of the triangle? **Activity** a) On a coordinate plane like that of figure 1, draw any triangle and measure the lengths of its sides. What can you say about the lengths of the sides of the triangle? Now draw a triangle with one side horizontal, and the other vertical. Is there anything more that you can say about the lengths of the sides.

Now, it is important to know that, by using a ruler we can always estimate the length of a line segment by a fraction (a rational number), and the accuracy of the estimate depends upon the detail of our ruler. The question we now want to raise is this: can any length be realized by a rational number?

*Construct square roots using tilted squares. Observe the pythagorean theorem in all of these*
tiled squares.

Explain a proof of the Pythagorean Theorem and its converse. (informally) 8G6

The coordinate system on the plane provides us with the ability to assign lengths to line segments. Let us review this:

![Figure 2](image)

**Example 2.** In Figure 2 we have drawn a tilted square (dashed sides) within a horizontal square. If each of the small squares bounded in a solid line is a unit square (the side length is one unit), then the area of the entire figure is $2 \times 2 = 4$. The dashed, tilted square is composed of precisely half (in area) of each of the unit squares, since each of the triangles outside the tilted square corresponds to a triangle inside the tilted square. Thus the tilted square has area 2 square units. Since the area of a square is the square of the length of a side, the length of each dashed line is a number whose square is 2; denoted $\sqrt{2}$.

We will use this symbol for every number: given a number $A$, that number whose square is $A$ is called the *square root* of $A$ and is denoted $\sqrt{A}$. We can, using the same strategy construct lengths corresponding to many square roots.

**Example 3.** In Figure 3 the large square has side length 3 units, and thus area of 9 square units. Each of the triangles outside the tilted square is a $1 \times 2$ right triangle, so is of area 1. Thus the area of the tilted square is 9-4 = 5, and the length of the sides of the tilted square is $\sqrt{5}$.

By the same reasoning: the large square in figure 4 is $7 \times 7$ so has area 49 square units. Each triangle outside the tilted square is a right triangle of leg lengths 3 and 4, so has area
6 square units. Since there are four of these triangles, this accounts for 24 square units, and thus the area of the tilted triangle is 49 - 24 = 25 square units. Since 25 = 5^2, the side of the tilted square has length 5 units. That is, √25 = 5. A positive integer whose square root is a positive integer is called a perfect square.

**Activity.** For any two lengths a and b, draw this figure:

Each of the triangles is a right triangle of leg lengths a and b; let c be the hypotenuse of that triangle. Then the tilted square has side length c, so has area c^2 square units. Now, cut out the four triangles and place them in the places shown in figure 5.

In figure 5, the area of the entire square is the sum of the area of the tilted triangle and the areas of the four triangles. In figure 6 the area of the entire square is the sum of the areas of the two smaller squares and the areas of the same four triangles. Since the entire squares in both figures are the same, this tells us that the area of the tilted square in figure 5 (c^2) is the sum of the e areas of the two squares (of side lengths a and b) shown in figure 6, which is a^2 + b^2. We conclude:

\[ c^2 = a^2 + b^2 \]
for a right triangle whose leg lengths are $a$ and $b$ and whose hypotenuse is of length $c$.

This is called the Pythagorean theorem, as it was attributed in the first treatise on geometry to the great geometer Pythagoras. We shall return to this theorem in Chapter 10.

The Pythagorean theorem allows us to find the lengths of the sides of tilted squares algebraically. For example, the tilted square in Figure 1 has side length $c$ where $c$ is the length of the hypotenuse of a right triangle whose leg lengths are both 1:

$$c^2 = 1^2 + 1^2 = 2,$$
so \( c = \sqrt{2} \). Similarly for figure 2: \( c^2 = 1^2 + 2^2 = 5 \), so \( c = \sqrt{5} \). For Figure 3, we calculate \( c^2 = 3^2 + 4^2 = 9 + 16 = 25 \), so \( c = \sqrt{25} = 5 \).

**Section 7.2. Rational and Irrational Numbers**

**Section 7.2a The Rational Number System**

*Understand informally that every number has a decimal expansion.*

*Know that numbers that are not rationals are called irrational. For rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.* 8NS1

The discussion about the relationship of numbers and lengths (summarized at the beginning of section 7.1), and their representation as decimals was a significant part of seventh grade mathematics. We now summarize that discussion. The decimal representation of rational numbers is the natural extension of the base ten place-value representation of whole numbers. Decimal fractions are constructed by placing a dot, called a *decimal point*, after the units’ digit and letting the digits to the right of the dot denote the number of tenths, hundredths, thousandths, and so on. If there is no whole number part in a given numeral, a 0 is usually placed before the decimal point (for example 0.75).

Thus, a decimal is a fraction whose denominator is not given explicitly, but is understood to be an integer power of ten. Decimal fractions are expressed using decimal notation in which the implied denominator is determined by the number of digits to the right of the decimal point. Thus for 0.75 the numerator is 75 and the implied denominator is 10 to the second power or 100, because there are two digits to the right of the decimal separator. In decimal numbers greater than 1, such as 2.75, the fractional part of the number is expressed by the digits to the right of the decimal value, again with the value of .75, and can be expressed in a variety of ways. For example,

\[
\frac{3}{4} = \frac{75}{100} = \frac{7}{10} + \frac{5}{100} = .75 ,
\]

\[
11 \quad 4 = 2 \frac{3}{4} = 2 + \frac{75}{100} = 2 + \frac{7}{10} + \frac{5}{100} = 2.75 .
\]

In seventh grade we observed that the decimal expansion of a rational number always either terminates after a finite number of digits or eventually begins to repeat the same finite sequence of digits over and over. Conversely such a decimal represents a rational number. First let’s look at the case of terminating decimals.

**Example 4.** First let us consider terminating decimals. Let us look at 0.275:

\[
0.275 = \frac{2}{10} + \frac{7}{100} + \frac{5}{1000} = \frac{2}{10} + \frac{7}{10^2} + \frac{5}{10^3} .
\]
Now, if we put these terms over a common denominator, we get

\[
\frac{2(10^2) + 7(10) + 5}{10^3} = \frac{275}{10^3}.
\]

In general, a terminating decimal is a sum of fractions, all of whose denominators are powers of 10. By multiplying each term by 10/10 as many times as necessary, we can put all terms over the same denominator. In the same way, .67321 becomes

\[
\frac{67321}{10^5},
\]

and

\[
0.0038 = \frac{3}{1000} + \frac{8}{10000} = \frac{38}{10000}.
\]

A terminating decimal leads to a fraction of the form \(A/10^e\) where \(A\) is an integer and \(e\) is a positive integer.

Notice that the expression \(A/10^e\) is not necessary in lowest terms:

\[
\frac{275}{10^3} = \frac{5 \cdot 5 \cdot 11}{2 \cdot 2 \cdot 5 \cdot 5 \cdot 5} = \frac{11}{2 \cdot 2 \cdot 5} = \frac{11}{40},
\]

and all we can say about this denominator is that it a product of 2’s and 5’s. But this is enough to guarantee that the fraction has a terminating decimal representation.

**Example 5.** a) \(25 = 5^2\), so we should expect that \(1/25\) can be represented by a terminating decimal. In fact, if we multiply by \(2^2/2^2\) we get:

\[
\frac{1}{25} = \frac{4}{100} = 0.04.
\]

b) Consider \(1/200\) by \(5/5\). Since \(200 = 2^3 \cdot 5^2\), we lack a factor of 5 in the denominator to have a power of 10. We fix this by multiplying the fraction by \(5/5\) to get \(5/1000 = .005\).

A terminating decimal leads to a fraction whose denominator is a product of 2’s and 5’s, and conversely, any such fraction is represented by a terminating decimal.

What about a fraction of the form \(p/q\), where \(q\) is not a product of 2’s and 5’s? In sixth grade we learned that by long division (of \(p\) by \(q\)) we can create a decimal expansion for \(p/q\) to as many places as we please. In seventh grade we went a little further. Since each step in the long division produces a remainder that is an integer less than \(q\) over a power of 10, after at most \(q\) steps we must repeat a remainder already seen. From that point on each digit of the long division repeats, and the process continues indefinitely in this way. For example, \(1/3 = .3333\ldots\) for as long as we care. For the division of 10 by 3 produces a quotient of 3 with a remainder of 1, leading to a repeat of the division of 10 by 3.
Example 6. Find the decimal expansion of $157/660$.

Solution. Dividing 157 by 660 gives a quotient of 2 and a remainder of 25.00. Divide 25.00 by 660 to get a quotient of .03 and a remainder of 5.200. So far we have

$$\frac{157}{660} = .23 + \frac{5.2}{660}.$$ 

Continuing division of the remainder by 660 produces a quotient of .007 and a remainder whose numerator is 58. Now division by 660 gives a quotient of .0008 and a remainder whose numerator is 52. Since that is what we had in the step that produced a 7, we’ll again get a quotient of 8 and a remainder of 58. Furthermore, these two steps continue to repeat themselves, so we can conclude that

$$\frac{157}{660} = .23787878 \cdots ,$$

with the sequence 78 repeating itself as often as we need. This will be written as $157/660 = .23\overline{78}$, where the over line indicates continued repetition.

Activity. Show that $73/275$ has the decimal expansion 0.2654. Find the decimal expansion for 1/7.

We conclude that

The decimal expansion of a fraction is eventually repeating - that is, after some initial sequence of digits, there a following set of digits that repeats over and over.

Section 7.2b Express Decimals as Fractions

To complete this set of ideas, we show that an eventually repeating decimal represents a fraction.

Example 7. Let’s start with: 0.33333\ldots, or in short notation $0.\overline{3}$. Let $a$ represent this number. Multiply by 10 to get $10a = 3.\overline{3}$. We then have the two equations:

$$a = 0.\overline{3}.$$ 

$$10a = 3.\overline{3}$$

Substitute $a$ for $0.\overline{3}$ in the second equation to get $10a = 3 + a$. We solve for $a$ to get $a = 0.\overline{3} = 1/3$. Subtract the second from the first to get $9a = 3$, to conclude that $a = 3/9 = 1/3$ is the fraction represented by the decimal.

Example 8. Here’s a more complicated example: $0.2\overline{34}$. Set $a = 0.2\overline{34}$. Now multiply by 1000, to get these two equations:

$$a = 0.2\overline{34}.$$ 

$$1000a = 234 + .\overline{23}$$
The second equation becomes (after substitution) \(1000a = 234 + a\), from which we conclude: \(999a = 234\), so \(a = \frac{234}{999}\), which, in lowest terms, is \(\frac{26}{111}\).

**Example 9.** We really should be discussing decimals that are eventually repeating, such as \(0.26\overline{54}\). First we take care of the repeating part: let \(b = 0.\overline{54}\), and follow the method of the preceding examples to get the equation \(100b = 54 + b\), so \(b = \frac{54}{99} = \frac{6}{11}\) in lowest terms. Now:

\[
0.26\overline{54} = \frac{26}{100} + 0.00\overline{54} = \frac{26}{100} + \frac{0.\overline{54}}{100} = \frac{26}{100} + \frac{6}{1000} = \frac{260 + 6}{1100} = \frac{292}{1100} = \frac{73}{275}.
\]

For a final example, let’s convert \(0.\overline{23}\). We have:

\[
0.\overline{23} = \frac{2}{10} + \frac{1}{10} \left(\frac{1}{3}\right) = \frac{7}{30}.
\]

**Section 7.2c Expand the Number System**

*Use square root and cube root symbols to represent solutions to equations of the form \(x^2 = p\) and \(x^3 = p\), where \(p\) is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that \(\sqrt{2}\) is irrational.* 8EE2

Every point on the real line gives a decimal expansion; at least as far as our measuring device goes. If we imagine that our measuring device can go as far as anyone ever will want, then we obtain an infinite decimal expansion. That is what \(0.\overline{3}\) means: no matter how many places we want in the decimal expansion, the digits are all 3’s.

We can describe the decimal expansion through measurement as follows: Let \(a\) be a point on the number line. Let \(N\) be the largest integer less than or equal to \(a\); that is \(N \leq a < N + 1\). Now divide the integer between \(N\) and \(N + 1\) into tenths, and let \(d_1\) be the number of tenths between \(N\) and \(a\). If this lands us right on \(a\), then \(a = N + d_1/10\). If not, divide the interval between \(N + d_1/10\) and \(N + (d_1 + 1)/10\) into hundredths and let \(d_2\) be the number of hundredths below \(a\), so that

\[
N + \frac{d_1}{10} + \frac{d_2}{100} \leq a < N + \frac{d_1}{10} + \frac{d_2 + 1}{100}.
\]

Now do the same thing with thousandths and continue indefinitely - or at least as far as your measuring device can take you.

Now, some decimal expansions are neither terminating, nor ultimately repeating, for example

\[
0.101001000100001000001 \cdots,
\]

where the number of 0’s between 1’s continues to increase by one each time, indefinitely. Another is obtained by writing down the sequence of positive integers right after each other:

\[
0.123456789101112131415 \cdots.
\]
Such numbers are called *irrational* numbers. We can continue to make up decimal expansions that are neither terminating nor repeating, and in that way illustrate more irrational numbers. But what is important to understand is that there are constructible lengths (like the sides of some tilted squares) that are irrational.

Let us return to section 7.1 and the discussion of square roots. The first few perfect squares are the squares of the integers: \(1, 4, 9, 16, 25, 36, 49, 64, 81, 100, \ldots\). So, 2, 3, 5, 6, 7, 8, and so forth are not perfect squares. [[Honors section: notice that the number of integers between successive perfect squares is 2, 4, 6, 8, 10... Does this pattern continue indefinitely?]] For numbers that are not perfect squares, we can sometimes use factorization to express the square root more simply.

**Example 10**. a) Since 100 is the square of ten, we can write \(\sqrt{100} = 10\). But we could also first factor 100 as the product of two perfect squares: \(\sqrt{100} = \sqrt{4} \cdot \sqrt{25} = 2 \cdot 5 = 10\).

b) Similarly, \(\sqrt{729} = \sqrt{9 \cdot 81} = \sqrt{9} \cdot \sqrt{81} = 3 \cdot 9 = 27\).

c) Of course not every number is a perfect square, but we still may be able to simplify: \(\sqrt{72} = \sqrt{36 \cdot 2} = \sqrt{36} \cdot \sqrt{2} = 6\sqrt{2}\).

d) \(\sqrt{32} = \sqrt{4 \cdot 4 \cdot 2} = 4\sqrt{2}\).

Similarly, we can try to simplify arithmetic operations with square roots:

e) \(\sqrt{6} \cdot \sqrt{12} = \sqrt{6} \cdot \sqrt{4} \cdot \sqrt{3} = \sqrt{6} \cdot 2 \cdot \sqrt{3} = 6\sqrt{3}\).

f) \(\sqrt{2} + \sqrt{8} = \sqrt{2} + \sqrt{4} \cdot \sqrt{2} = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}\).

We know that a rectangle in three dimensions has volume \(abc\) cubic units, where \(a, b, c\) are the lengths of the sides of the rectangle. In particular, if a cube has side length \(a\), then its volume is \(a^3\). If we are given the volume of the cube, \(V\), then the side length has to be a number whose cube is \(V\); this is called the *cube root* of \(V\), and is denoted \(\sqrt[3]{V}\). The first few perfect cubes are the cubes of the integers: 1, 8, 27, 64, 125, and so forth.

Now, the fact that we are heading for is this: for any integers \(N\), if \(N\) is not a perfect square, then \(\sqrt{N}\) is not expressible as a fraction; that is, it is an irrational number. The existence of irrational numbers was discovered by the ancient Greeks (about 5th century BCE), and they were terribly upset by the discovery, since it was a basic tenet of theirs that numbers and length measures of line segments were different representations of the same idea. What happened is that a member of the Pythagorean society showed that it is impossible to express the length of the side of the dashed square in figure 2 (\(\sqrt{2}\)) by a fraction. Here we’ll try to describe the modern argument that actually proves more:

**If \(N\) is a whole number then either it is a perfect square or \(\sqrt{N}\) is irrational**

Another way of making this statement is this: If \(N\) is a whole number, and \(\sqrt{N}\) is rational, then \(\sqrt{N}\) is also a whole number. So, let’s start with the integer \(N\), and suppose that
\[ \sqrt{N} = \frac{p}{q}, \text{ where } p \text{ and } q \text{ are whole numbers. We can suppose that the fraction } \frac{p}{q} \text{ is in lowest terms.} \]

To verify the statement, we need to show that \( q \) is 1. To do this, we first rewrite the equation \( \sqrt{N} = \frac{p}{q} \): by squaring both sides we get \( N = \frac{p^2}{q^2} \), and then multiplying by \( q^2 \) we get to

\[ p^2 = Nq^2. \]

Suppose that \( q \) is even. Then \( q^2 \) is even, so the right hand side is even, and thus \( p^2 \) is also even. Since the square of an odd number is odd, that tells us that \( p \) is also even. But it can’t be that both \( p \) and \( q \) are even, for the fraction \( \frac{p}{q} \) is in lowest terms, meaning that \( p \) and \( q \) cannot both be divisible by 2.

So, \( q \) is not divisible by 2. Now, this argument works for any prime: if a prime number divides \( q \), then also has to divide \( p \) (try 3, 5, 7). This tells us that the whole number \( q \) has no prime factors; but the only whole number with no prime factors is 1, so \( q = 1 \), which tells us that \( \sqrt{N} \) is a whole number.

**Example 11.** The square root of 50 is irrational. For \( 50 > 7^2 \) and \( 50 < 8^2 \), so \( \sqrt{50} \) lies between 7 and 8, and thus cannot be an integer. Another way to see this is to factor 50 as \( 25 \cdot 2 \), from which we conclude that \( \sqrt{50} = 5\sqrt{2} \), and \( \sqrt{2} \) is irrational.

**Example 12.** \( \pi \) is another irrational number that is is defined geometrically. How do we give a value to this number? The ancient Greek mathematician, Archimedes, was able to provide a geometric way of approximating the value of \( \pi \). Here we describe it briefly, using its definition as the quotient of the circumference of a circle by its diameter. In figure 8, consider the triangles drawn to be continued around the circle 8 times.

![Figure 6](image)

Now measure the lengths of the diameter (5.3 cm) and the lengths of the outer edge of the triangles (2, 2.3). Then, once the triangles are reproduced in each half quadrant, we obtain
two octagons, one with perimeter $8 \times 2$, which is less than the circumference of the circle and the other with perimeter $8 \times 2.3$ which is greater than the circumference of the circle. Thus

$$\frac{8 \times 2}{5.3} < \frac{C}{D} = \pi < \frac{8 \times 2.3}{5.3}.$$ 

After executing the division we obtain this estimate: $3.02 < \pi < 3.47$. To increase the accuracy, we increase the number of sides of the approximating polygon. Archimedes created an algorithm to calculate the polygon circumferences each time the number of sides is doubled, and at the next step (a polygon with 16 sides) produced the estimate $\frac{22}{7}$ for $\pi$; which is correct within an error of 0.002.

Section 7.2d Approximating the value of irrational numbers

Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., $\pi^2$). 8NS2

Since irrational numbers are represented by decimals that are neither terminating nor repeating, we have to rely on the definition of the number to find fractions that are close to the number - hopefully as close as we need the approximation to be. So, as we saw above, Archimedes used the definition of $\pi$ to find a rational number ($\frac{22}{7}$) that is within 1/500 of $\pi$. Even better, Archimedes described an algorithm to get estimates that are closer and closer - as far as we need them to be. But how about square roots?

We know that $\sqrt{2}$ can be represented as the diagonal of a right triangle with leg lengths equal to 1, so we can measure the length with a ruler. But the accuracy of that measure depends upon the detail in our ruler. We’d rather have an arithmetic way to find square roots - or, to be more accurate, to approximate them. For this, we return to the discussion of decimals in 7th grade. There a method was described that found decimals that came as close as possible to representing a given length. However, it too, depended upon our capacity to measure. What we want is a method to approximate square roots, which, if repeated over and over again, gives us an estimate of the square root that comes as close to the exact length as we need it to be.

Before calculators were available, a method for estimation of square roots was that of trial and error. With calculators, it is not so tedious - let us describe it.

Example 13. Approximate $\sqrt{2}$ correct to three decimal places.

Solution. Since $1^2 = 1$ and $2^2 = 4$, we know that $\sqrt{2}$ is between 1 and 2. Let’s try 1.5. $1.5^2 = 2.25$, so 1.5 is too big. Now $1.4^2 = 1.96$ so 1.4 is too small, but not by much: 1.4 is correct to one decimal place. Let’s try 1.41: $1.41^2 = 1.9881$ and $1.42^2 = 2.0164$. It looks like 1.42 is a little closer, so let’s try 1.416: $1.416^2 = 2.0055$; so we try 1.415: $1.415^2 = 2.002$, still a little large, but quite close. To see that we are within three decimal places, we check $1.414^2 = 1.9994$. That is pretty close, and less than 2. To check that 1.414 is correct to three
decimal places, we check half an additional point upwards: \(1.4145^2 = 2.0008\), so the exact value of \(\sqrt{2}\) is between 1.4140 and 1.4145, so 1.414 is correct to three decimal places.

**Example 14.** Gregory has a square plot of land behind his house of 2000 sq.ft. and he wants to divide it into 4 square plots of equal area, in which he will plant different vegetables. What should be the side length of each plot?

**Solution.** First we try to get close: The square of 30 is 900; so we try 40: its square is 1600, and 50\(^2 = 2500\), so our answer is between 40 and 50. We try 45; since 45\(^2 = 2025\), we are close. This may satisfy Gregory: make each plot a little smaller than a 45 by 45 foot square. But Gregory persists: he calculates 44\(^2 = 1936\), leading to the guess that the good measure of a side is between 44 and 45 - and maybe closer to 45. So Gregory calculates the squares of number between 45.5 and 50:

<table>
<thead>
<tr>
<th>Number</th>
<th>Square of number</th>
</tr>
</thead>
<tbody>
<tr>
<td>44.5</td>
<td>1980.25</td>
</tr>
<tr>
<td>44.6</td>
<td>1989.16</td>
</tr>
<tr>
<td>44.7</td>
<td>1998.09</td>
</tr>
<tr>
<td>44.8</td>
<td>2007.04</td>
</tr>
</tbody>
</table>

and concludes that each plot has to have side length between 44.7 feet and 44.8 feet, and much closer to 44.7; He settles on a side length of 44.72, and does his planting. How close did he get? The square of 44.72 (accurate to two decimal points is 1999.88 - not bad.

**Example 15.** Find the square root of 187 accurate to two decimal points.

**Solution.** Since \(10^2 = 100\), we try \(11^2 = 121\), \(12^2 = 144\), \(13^2 = 169\) and \(14^2 = 196\), and conclude that \(\sqrt{187}\) is between 13 and 14, and probably a little closer to 14. So, try 13.7: 13.7\(^2 = 187.69\), a little too big. Now try 13.65: 13.65\(^2 = 187.005\); almost there! Now, to be sure we are correct to 2 decimal points, we calculate 13.64\(^2 = 186.05\). Since 13.65\(^2\) is so much closer to 187, we conclude that 13.65 is the answer, accurate to two decimal points.

Trial and error seems to work fairly well, so long as we start with a good guess, and have a calculator at hand. But a mathematician seeks a method that always works and doesn’t depend upon good guesses. There is one, called “Newton’s method,” after Isaac Newton, one of the discoverers of the Calculus. Newton reasoned this way: Suppose that we have an estimate \(e\) for the square root of \(N\), That is: \(e^2 \sim N\), where the symbol \(\sim\) means “is close to.” Dividing by \(e\), we have \(e \sim N/e\), so \(N/e\) is another estimate; about as good as \(e\). He also noticed that \(e\) and \(N/e\) lie on opposite sides of \(N\), so their average should be an even better estimate. So he set

\[
e' = \frac{1}{2}(e + \frac{N}{e}),
\]

and then repeated the logic with \(e'\), and once again with the new estimate, until the operation of taking a new average did produced the same answer, up to the desired number of decimal points.
Example 16. Now find the square root of two accurate up to four decimal places.

Solution. We try 1, since \(1^2 = 1\), and \(2^2 = 4\), and so 1 seems to be the closest integer. Now \(2/1 = 2\), so according to Newton, we should now try the average: 1.5. Since \(2/(1.5) = 4/3\), we next try the new average:

\[e' = \frac{1}{2}(1.5 + \frac{4}{3}) = \frac{17}{12} - 1.4167.\]

Now, this is where calculators or a computational program like excel come in: we can just repeat the process until it stabilizes. Look at Figure 6 below; these are excel calculations using Newton’s method. Start with the first table. Our discussion brought us to the second line (1.416666667 is 17/6 correct to 9 decimal places). The table continues with Newton’s method, until the estimate (last column) stabilizes. Actually, it stabilized (to four decimal places) in the next line, but we have continued the calculation, to show that there is no further change. In fact each step in Newton’s method produces a more accurate estimate, so once we have no change (to the number of correct decimals we require), there is no need to go further. The remaining tables of Figure 6 are for \(\sqrt{5}\), \(\sqrt{25}\), \(\sqrt{1000}\). Note how quickly the estimate for \(\sqrt{25}\) settles on 5. In the following example, we describe the content of these tables.

Example 17. Use Newton’s method to find the square root of 5.

Solution. First of all “find the square root of five” is not very meaningful. Using the tilted square with side lengths 1 and 2, we found the square root of 5 as a length. So, perhaps here wen mean, ”find the numerical value of the square root of 5.” But as we have already observed, \(\sqrt{5}\) is not expressible as a fraction, so we can’t expect to “find” its value precisely. What we can hope for is to find a decimal expansion that comes as close as we please to \(\sqrt{5}\). So, let us make the question precise: find a decimal that is an estimate of \(\sqrt{5}\) that is correct to 4 decimal places.

First we see that 2 is a good approximation for \(\sqrt{5}\) by an integer, since \(2^2 = 4\) and \(3^2 = 9\). Now \(5/2 = 2.5\) is also a good guess, since \(2.5^2 = 6.25\). Newton’s method asks us now to try the average of 2 and 2. \(2.25\). Now \(5/2.25 = 2.222\ldots\), and the average of 2.25 and \(5/2.25\) is 2.361. To see if this is a good approximation of \(\sqrt{5}\) we calculate \(2.361^2 = 5.5731\). We’re closing in. Now lets go one more step. Start with the last “best guess,” 2.361. \(5/2.361 = 2.118\), and the average of these two numbers is 2.2395, and the square of 2.2395 is 5.0154. That is really good! One more time: Take the average of 2.2395 and 5/2.2395 = 2.2362; it is 2.2361. We check to see how close we are: 2.2361\(^2\) = 4.999\ldots\) We can feel secure that 2.2361 is, up to three decimal places, the square root of 5. We can continue this process for as long as we need to to arrive at the desired precision. If we need 4, 5 or 10 decimals of precision, we continue this process until two last calculated approximations agree up to the desired number of decimal places. Figure 7 shows us that by the third step in Newton’s method, we have \(\sqrt{5}\) correct to 6 decimal places.

Do we ever get to the place where the “last two approximations” agree perfectly - that is: when have we arrived at the numerical value of the desired point? Alas, the answer s ”we
Figure 7

<table>
<thead>
<tr>
<th>Estimate</th>
<th>2/e</th>
<th>New Estimate (Average)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>1.3333333</td>
<td>1.416666667</td>
</tr>
<tr>
<td>1.41667</td>
<td>1.411765</td>
<td>1.41215686</td>
</tr>
<tr>
<td>1.41422</td>
<td>1.414211</td>
<td>1.414213562</td>
</tr>
<tr>
<td>1.41421</td>
<td>1.414214</td>
<td>1.414213562</td>
</tr>
<tr>
<td>1.41421</td>
<td>1.414214</td>
<td>1.414213562</td>
</tr>
<tr>
<td>1.41421</td>
<td>1.414214</td>
<td>1.414213562</td>
</tr>
<tr>
<td>1.41421</td>
<td>1.414214</td>
<td>1.414213562</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Estimate</th>
<th>5/e</th>
<th>New Estimate (Average)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.25</td>
<td>2.2222222</td>
<td>2.236111111</td>
</tr>
<tr>
<td>2.23611</td>
<td>2.236025</td>
<td>2.236067978</td>
</tr>
<tr>
<td>2.23607</td>
<td>2.236068</td>
<td>2.236067977</td>
</tr>
<tr>
<td>2.23607</td>
<td>2.236068</td>
<td>2.236067977</td>
</tr>
<tr>
<td>2.23607</td>
<td>2.236068</td>
<td>2.236067977</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Estimate</th>
<th>730/e</th>
<th>New Estimate (Average)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>30.76923</td>
<td>31.63461538</td>
</tr>
<tr>
<td>31.6346</td>
<td>31.61094</td>
<td>31.62277882</td>
</tr>
<tr>
<td>31.6228</td>
<td>31.62277</td>
<td>31.6227766</td>
</tr>
<tr>
<td>31.6228</td>
<td>31.62278</td>
<td>31.6227766</td>
</tr>
<tr>
<td>31.6228</td>
<td>31.62278</td>
<td>31.6227766</td>
</tr>
</tbody>
</table>

never do, but we do get better and better.” Unless N itself is a perfect square (the square of another integer), we will never arrive at a decimal expansion that is precisely \( \sqrt{N} \). But - and this is all we really need - we can get as close as we want.

**Activity.** Now use Newton’s method to calculate \( \sqrt{2000} \), \( \sqrt{187} \).

**Activity.** It is not necessary that the first guess at the square root is close to the answer - we can start with any positive number and end up with the estimate we want - it just may take a little longer. To see this, use Newton’s method to find \( \sqrt{150,000} \), starting with 2 as the first guess. See table 7.

[[Honors: to find cube roots, Newton’s method suggest using the recursion \( e' = \frac{1}{3}e + \frac{2N}{3e^2} \). It is good, but we need a reason why we do that.]]

We can extend arithmetic relations and operations to irrational numbers since the decimal expansion allows us to get as close as we please to any number.

**Example 18.** What is bigger \( \pi^2 \) or 10? To answer such a question, we need to find a rational number larger than \( \pi \) whose square is smaller than 10. 3.15 will do, since it is larger than \( \pi \) and \( 3.15^2 = 9.9225 < 10 \).

But we do have to be a little careful: the closeness of approximations changes as we add or
### Square root of 150000

First estimate:  

<table>
<thead>
<tr>
<th>Estimate</th>
<th>$150000/e$</th>
<th>New Estimate (Average)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>150000</td>
<td>75000.5</td>
</tr>
<tr>
<td>75000.5</td>
<td>1.999987</td>
<td>37501.24999</td>
</tr>
<tr>
<td>37501.2</td>
<td>3.999867</td>
<td>18752.62493</td>
</tr>
<tr>
<td>18752.6</td>
<td>7.99888</td>
<td>9380.311905</td>
</tr>
<tr>
<td>9380.31</td>
<td>15.99094</td>
<td>4698.151422</td>
</tr>
<tr>
<td>4698.15</td>
<td>31.92745</td>
<td>2365.039437</td>
</tr>
<tr>
<td>2365.04</td>
<td>63.42389</td>
<td>1214.231663</td>
</tr>
<tr>
<td>1214.23</td>
<td>123.5349</td>
<td>668.8832869</td>
</tr>
<tr>
<td>668.883</td>
<td>224.2544</td>
<td>446.5688284</td>
</tr>
<tr>
<td>446.569</td>
<td>335.8945</td>
<td>391.2316493</td>
</tr>
<tr>
<td>391.232</td>
<td>383.4046</td>
<td>387.3181067</td>
</tr>
<tr>
<td>387.318</td>
<td>387.2786</td>
<td>387.2983351</td>
</tr>
<tr>
<td>387.298</td>
<td>387.2983</td>
<td>387.2983346</td>
</tr>
<tr>
<td>387.298</td>
<td>387.298</td>
<td>387.2983346</td>
</tr>
</tbody>
</table>

**Figure 8**

multiply them. So, if $a$ agrees with $a_0$ up to two decimal points, and $b$ agrees with $b_0$ up to two decimal points, we cannot conclude that $a + b$ or $ab$ agree with $a_0 + b_0$ or $a_0 b_0$ up to two decimal points. Let us illustrate that.

**Example 19.** $3.16$ agrees with $\sqrt{10}$ to two decimal points. But $3.16 \times 3.16 = 9.9856$, which does not agree with 10 to two decimal points.

**Example 20.** Approximate $\pi + \sqrt{2}$ to three decimal points. Start with the approximations up to four decimal points: $3.1416$ and $1.4142$, and add: $3.1416 + 1.4142 = 4.5558$, from which we accept $4.556$ as the three decimal approximation.

For products it is not so easy. Suppose that $A$ and $B$ are approximations to two particular numbers, which we can denote as $A_0, B_0$, Then we have $A_0 = A + e$, $B_0 = B + e'$ where $e$ and $e'$ are the errors in approximation. For the product we will have $A_0 B_0 = (A + e) (B + e') = AB + Ae' + Be + ee'$, telling us that the magnitude of the error has been multiplied by the factors $A$ and $B$, which may put us very far from the desired degree of accuracy.

**Example 21** Suppose that $A$ and $B$ approximate the numbers 1 and 100 respectively within one decimal point. That means that

$$0.95 < A < 1.05 \quad \text{and} \quad 99.95 < B < 100.05.$$  

When we multiply, we find that $AB$ could anywhere in the region

$$94.9525 < AB < 105.0525.$$
thus, as much as 5 units away from the accurate product 100.

To find approximate values for irrational numbers, we have to understand the definition of the number so we can use it for this purpose. For example, suppose we want to find a number whose cube is 35, correct up to two decimal places. Start with a good guess. Since $3^3 = 27$ and $4^3 = 64$, we take 3 as our first guess. Since 27 is much closer to 35 than 64, we now calculate the cubes of 3.1, 3.2, 3.3, ... until we find those closest to 30: $3.1^3 = 29.791$, $3.2^3 = 32.6768$, $3.3^3 = 35.937$. We can stop here, since the last calculated number is larger than 35. Now we go to the next decimal point, starting at 3.3, working down (since 35.937 is closer to 35 than 32.6768: $3.29^3 = 35.611$, $3.28^3 = 35.287$, $3.27^3 = 34.965$. Since the last number is as close as we can get to 3.5 with two place decimals, we conclude that 3.27 is correct to two decimal places.