Chapter 4. Functions

We start this chapter with an important change in our understanding of the use of letters $x, y, z$, to represent unknown numbers or quantities. In both those cases we think of $x$ (or $y$ or $z$ or ... ) as a yet-to-be-determined number (or numbers) to be found by “solving” in the case of “unknowns”, and “measuring” in the case of “quantities.” But now we start to interpret the symbols $x, y, z$ as variables, that is, they are to be understood as ranging over a whole set of numbers, and we study how the relation among those variables affects their variation. As we shall see, this is not so hard if the relation is expressed as a graph, harder if expressed algebraically or by a table, and difficult if expressed by an algorithm. Our interest changes from that of solving equations to that of understanding the relations among the variables expressed by these equations so that we can make predictions about the relationship in the future (or in the past). In a more important sense, we have moved from a “static” study of relations to a dynamic one; it is in this sense that letters represent “variables.”

An equation with two variables $x, y$ expresses a relationship between them. A solution of the equation consists of two specific numbers, one for each variable, which, when substituted in the equation makes a true statement. In case there is more than one solution, we may talk about the solution set. We usually use an ordered pair $(x, y)$ to represent each solution. The order indicates which variable represent which number. Thus, the instruction “substitute $(5, -1)$ in the equation” means: set $x = 5$ and $y = -1$. For example, if the relation is $3x - 2y = 1$, then $(1, 1)$ is in the relation, but $(2, 3)$ is not.

Suppose that we are given a relation between $x$ and $y$. It may happen that, when a specific number is substituted for $x$, we can solve the equation for $y$. In this case we say that $y$ is a function of $x$. More generally, any set of instructions that produces an output number corresponding to an input number is a function. The set of instructions is not the focus of our study any more, our analysis will be on the relation of inputs to outputs. For this reason, it is customary to refer to a function as a black box (see figure 1), to stress that the concern is with input/output pairs.

![Figure 1](image)

The core of this chapter develops the theme of moving from the concept of unknown to that of variable, and that of studying the relationships among variables. We will focus on characteristics that separate linear from nonlinear functions. In the last section we introduce techniques for discerning the behavior of the values of a function with respect to the input variable through the various representations of a function.
Section 4.1. What is a Function?

Understand/identify functions. Compare properties of two functions each represented in a different way (tables, graphs, equations). 8F1,2

Example 1. The following table is that of the bus schedule between Salt Lake City and Price.

<table>
<thead>
<tr>
<th></th>
<th>Salt Lake City to Price</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Lv SLC</td>
</tr>
<tr>
<td></td>
<td>Arr Price</td>
</tr>
</tbody>
</table>

Examining the table, we notice several things: first of all, it takes the 8:00 bus three hours 15 minutes to make the trip; furthermore, this time is the time every trip takes. Also, the change between any two departure times is the same as the change between any two arrival times. From these data, that the rate of change of arrival time with respect to departure time is constant, we conclude that the graph is a straight line. In fact, here is the graph:

![Graph showing bus schedule](image)

Figure 2

We conclude that, given any departure time from Salt Lake City, the bus arrives in Price three hours, 15 minutes later. So, if a new bus, with departure time 6:30 were to be added to the schedule, we should schedule it to arrive in Price at 9:45. More generally, if a bus leaves SLC at $D$ o’clock, it should be expected to arrive in Price at $D + 3 : 15$ o’clock. Letting $A$ represent the arrival time, we arrive at this relationship between $D$ and $A$: $A = D + 3 : 15$. We see that this formula tells us that the arrival time is completely determined by the departure time; that is $A$ is a function of $D$. In such a statement, we consider $A$ and $D$ as variables in the sense that they can have any (time) value, and the relation $A = D + 3 : 15$ will hold.

Before going on, we should observe that, in the real world, arrival time is not completely determined by departure time; factors along the road may delay, or advance, the arrival of the bus. The following is a more realistic graph of what may actually happen in a day.
This graph, of actual data, does give us important information (if we did not have a schedule handy): we should expect, on average, for the trip to take 3 hours and 15 minutes. However, in the early morning and late evening, the trip is likely to be quicker, while in the late afternoon, it is likely to take longer. We shall return to the contrast between real and simulated data in Chapter 6. Our goal there will be to interpret tables of actual data so as to discover an actual formula that best relates the variables. But for now, let us consider relations between two variables that give rise to functions.

**Figure 3**

Given two variables, $x$ and $y$, we will say that $y$ is a function of $x$ if there is a rule (formula, algorithm or set of instructions) that determine $y$ for a given $x$.

**Examples**

1. $y = 3x + 7$.
   This can also be given by the set of instructions: pick a number $x$, multiply it by 3 and add 7. Notice that the instructions clarify the order of operations much better than the formula does, so it is good practice to translate formulas to sets of instructions, as well as going from instructions to formulas, especially when using a calculator to execute the operations.

2. $y = x^2 - x + 1$.
   Here the instructions are: pick a number, $x$. Square it to get $x^2$. Subtract $x$. Add 1.

3. $y = 1/x$.
   Here, we do not have a rule to give a value of $y$ corresponding to $x = 0$. We say that the function is not defined for $x = 0$, or $x$ is not in the domain of the function. For this function, in most contexts, $x$ is a positive number, so we often make explicit that we are only interested in the function for positive values of $x$. For this function, we say that $x$ and $y$ are inversely proportional in the sense that if $x$ is multiplied by any number, the $y$ is divided by that number.

4. $y - 2x = 11$.
   Sometimes the relation between the variables is given in a way that some algebraic operations must be performed in order to find the rule. Here it is easy: add $2x$ to both sides of the equation to get $y = 2x + 11$.

5. $y^2 = 1 - x$. 
This is a formula relating $y$ and $x$, but does not describe $y$ as a function of $x$. If we take $x = 0$, the output $y$ could be $+1$ or $-1$, so there is no unique value of $y$ corresponding to this $x$. This is true for all values of $x$ less than 1, and furthermore, there are no values of $y$ corresponding to any $x$ greater than 1. So, we do not have a set of rules that unambiguously produce a unique $y$ for any given $x$, and thus we do not have a function. Notice that by adding $x$ to both sides of the equation, and subtracting $y^2$ from both sides of the equation we can write $x$ as a function of $y$: $x = 1 - y^2$.

Plotting a set of values $(x, y)$ that are related by a function provides a useful visualization of the function. The usefulness depends upon the extent to which the selected points illustrate the important features of the function. So, given rules describing a function, we calculate a set of points $(x, y)$, where, for an arbitrary value $x$, $y$ is the function value corresponding to $x$. When we plot enough points, we join them with a curve to get a representation of the function. For the general function this may take some skill or additional information contained in the context, but - as we have seen in the preceding chapter - for a linear function we need only find two points on the graph, and connect them with a line.

Let’s go through this analysis for each of the above examples.

1. $y = 3x + 7$
   Make a table of representative values

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>1</td>
<td>4</td>
<td>7</td>
<td>10</td>
<td>13</td>
<td>16</td>
<td>19</td>
</tr>
</tbody>
</table>

   ![Figure 4](image)

2. $y = x^2 - x + 1$

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>7</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>
3. $y = 1/x$.
The rule here is “pick a positive number and take its inverse.” We create the table using the first 8 positive half integers:

<table>
<thead>
<tr>
<th>$x$</th>
<th>.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>2</td>
<td>1</td>
<td>.67</td>
<td>.5</td>
<td>.4</td>
<td>.33</td>
<td>.29</td>
<td>.25</td>
</tr>
</tbody>
</table>

4. $y - 2x = 11$
Rewrite as $y = 2x + 11$ and create a table like this:

<table>
<thead>
<tr>
<th>$x$</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>13</td>
<td>15</td>
<td>17</td>
<td>19</td>
</tr>
</tbody>
</table>

Notice that every time $x$ increases by 1, $y$ increases by 2. Recall that this tells us that 2 is the slope of the line, or the unit rate of change of $y$ with respect to $x$. 
5. $y^2 = -x + 1$ First, we note that it is easier to make the table by writing the relation in the form $x = 1 - y^2$ and finding values of $x$ corresponding to values of $y$. This gives us the table and graph

<table>
<thead>
<tr>
<th>$x$</th>
<th>-8</th>
<th>-3</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>-3</th>
<th>-8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>-3</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Figure 7

We see from this graph that this does not specify $y$ as a function of $x$; at least not until we include a rule that tells us, for any $x$ which of the two candidate values is to be chosen. We may, for example, add the rule: For each $x$, let $y$ be the positive number such that $y^2 = -x + 1$. Then we get this graph, which now describes a function:
To summarize: figure 8a (of the relation $y^2 = -x + 1$) does not describe $y$ as a function of $x$, because of the ambiguity in taking the positive or the negative solution of $y^2 = x + 1$. This ambiguity is resolved by adding the stipulation: For a given $x < 1$, let $y$ be the positive solution of the equation $y^2 = x + 1$, resulting in figure 8b.

**Functions defined by graphs**

A graph can define a function using this as the rule:

> Given a value for $x$, draw the vertical line through that value on the $x$-axis. Where it hits the graph, draw the horizontal to the $y$-axis. That point is the value of $y$ corresponding to the given value of $x$.

For this rule to work, we must know two things:

a) for a given number $a$, the vertical line $x = a$ intersects the graph;

b) for a given number $a$, the vertical line $x = a$ intersects the graph only once.

If these two conditions are satisfied, then the rule works: the vertical line through $a$ intersects the graph at one point, and the horizontal line through that point intersects the $y$ axis at some point $b$. This $b$ is the value of the function for the input $a$.

If either condition fails for a number $a$, then the function cannot be defined at $a$. We express this by saying that $a$ is not in the *domain* of the function. Just to say this another way, given a graph it defines a function for all numbers $a$ for which this condition holds. That set of numbers is the *domain* of the function, and for any $a$ in the domain, the above rule produces the value of the function at $a$. 

**Figure 8b**
Examples. For the following graphs, make a table of values of the function by applying the above rule.

![Graph 1]

Applying the rule, we can create this table of values of the function:

<table>
<thead>
<tr>
<th>$x$</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>11</td>
<td>8</td>
<td>5</td>
<td>2</td>
<td>-1</td>
<td>-4</td>
<td>-7</td>
</tr>
</tbody>
</table>

Given the shape of the graph, we can assume that this is a straight line. In this case we can just pick two points to get the equation of the line. For example, pick $(-2, 8)$ and $(1, -1)$, and calculate the slope:

$$m = \frac{8 - (-1)}{-2 - 1} = \frac{9}{-3} = -3.$$

Since the $y$-intercept is given by the point $(0, 2)$, we know that $b = -2$, so the equation for the function is $y = -3x + 2$.

![Graph 2]

<table>
<thead>
<tr>
<th>$x$</th>
<th>-1</th>
<th>-.6</th>
<th>-.3</th>
<th>0</th>
<th>.2</th>
<th>.5</th>
<th>.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0</td>
<td>.8</td>
<td>.95</td>
<td>1</td>
<td>.98</td>
<td>.87</td>
<td>.6</td>
</tr>
</tbody>
</table>
In the first example, because the graph went through integer points, the pairs \((x, y)\) were easy to find according to the set of rules for graphs. In this example we have to pick values of \(x\) for which we could most easily estimate values of \(y\). In any case, this is not a line.

**Activity.** Create tables of values for the following graphs.

a)

b)

---

**Section 3.2. Linear and Nonlinear Functions**

*Interpret the equation \(y = mx + b\) as a linear function. Observe that if \(b\) is not zero, the variables are not in proportion; however, the change is in the variables between two points are in proportion (hence the idea of slope). 8F3*

*Distinguish between linear and nonlinear functions.*

The defining characteristic of a line is this: for any two points \(P\) and \(Q\), the ratio of the change in \(y\) from \(P\) to \(Q\) to the change in \(x\) is a constant, called the slope of the line and denoted by \(m\).*
Important things to remember are:

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intersects y-axis at (0, b)</td>
<td>( y = mx + b )</td>
</tr>
<tr>
<td>Horizontal</td>
<td>( y = b )</td>
</tr>
<tr>
<td>Vertical</td>
<td>( x = a )</td>
</tr>
<tr>
<td>Through origin</td>
<td>( y = mx ) and ( y/x = m )</td>
</tr>
<tr>
<td>Slope ( m ), point ((x_0, y_0))</td>
<td>( y - y_0 = m(x - x_0) )</td>
</tr>
<tr>
<td>((x_0, y_0), (x_1, y_1))</td>
<td>( \frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0} )</td>
</tr>
</tbody>
</table>

**Activity 1.** For each of the following functions, determine whether or not the graph is a straight line. In each case, draw the graph of the equation.

a) \( y = 3x + 1 \)

b) \( y = 3(x + 1) \)

c) \( y = x(x + 1) \)

d) \( y = -2x + 1 \)

e) \( y = -2(x + 1) \)

f) Given any input \( x \), the output \( y \) is the largest integer less than \( x \).

**Activity 2.** For each of the following tables, determine whether or not the points \((x, y)\) lie on a line. In each case graph the points in the table and connect them with a curve that is as close as possible to a line.

a)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
</tr>
</tbody>
</table>

b)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-4</td>
<td>-1</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>11</td>
</tr>
</tbody>
</table>
c) 
\[ \begin{array}{cccccc}
    x & 0 & 1 & 2 & 3 & 4 & 5 \\
    y & 7 & 11 & 14 & 18 & 22 & 24 
\end{array} \]

d) 
\[ \begin{array}{cccccc}
    x & 0 & 1 & 2 & 3 & 4 & 5 \\
    y & 6 & 4 & 2 & 0 & -2 & -4 
\end{array} \]

e) 
\[ \begin{array}{cccccc}
    x & 0 & 1 & 2 & 3 & 4 & 5 \\
    y & 10 & 9 & 7 & 4 & 0 & -5 
\end{array} \]

f) 
\[ \begin{array}{cccccc}
    x & 0 & 1 & 2 & 3 & 4 & 5 \\
    y & 2 & 2 & 2 & 2 & 2 & 2 
\end{array} \]

**Section 4.3. Modeling and Analyzing a Functional Relationship**

This section - like many of the topics in 8th grade - is exploratory. The purpose here is twofold. First, to decide, for a given context, if it suggests a functional relationship between two variables, and in particular, when can we conclude that it a linear function? The second purpose is to try to figure out the properties of a functional relationship from its rules, a table of corresponding values, or a graph.

We begin by illustrating, through examples, contexts that lead to a functional relationship. We will concentrate on the determination of a linear relationship either from the context, or the data that have been gathered. In many cases, as in our first few examples, the context clearly indicates a constant rate of change, and thus, a linear relationship. The subsequent examples show a variable rate of change; here we explore what the data can tell us.

**Constructing Functions**

*Construct a function to model a linear relationship. Determine rate of change, initial value (use representations + context). 8F4*

**Example 1. Heat and Temperature.** The temperature of an object measures the amount of heat it contains. Temperature is measured in degrees, denoted \(^\circ\)C. No matter what scale is used, it should be such that a change in the temperature of an object is proportional to the change in heat, as measured by caloric content. Otherwise put, temperature is a linear function of caloric content. However, caloric content is hard to measure directly, and so we turn to other means to quantify heat. For example, some fluids expand in volume as they heat up, and in a linear way: the change in volume is proportional to the change in caloric content. Mercury is such a fluid, and thus is the fluid of choice in a thermometer. As the object heats up, the mercury expands and the column of fluid in the stem of the thermometer rises. The important thing is that this change in height of the column of mercury is proportional to the change in volume, and thus proportional in a change in heat.

The Celsius temperature scale is based on water: 0\(^\circ\)C corresponds to the heat content of newly melted ice, and 100\(^\circ\)C to water just starting to boil. Thus, if a pot of water measures 50\(^\circ\)C, the increase in caloric content from 0\(^\circ\)C is half the increase in caloric content of the same amount of water at the boiling point. Daniel Gabriel Fahrenheit was a doctor in the 18th century who wanted to measure the heat generated by a disease in a human patient, so he invented a scale that was based on humans: 100\(^\circ\)F is the temperature of a healthy human being, and 0\(^\circ\)F is the temperature of blood just about
to freeze. So, for example, if a person shows a temperature of 102°F, that person is 2% hotter than a healthy person. By experimentation, Fahrenheit discovered that, in his temperature scale, the freezing point of pure water is 32°F, and the boiling point of water is 212°F.

Activities
1. Given that these two temperature scales are linear with respect to caloric content, they are linear with respect to each other. So we can relate °C with °F by a linear relation. We know two points on the graph of this relation: the freezing point of water, (0, 32) and the boiling point of water, (100, 212) (where we have put °C as the first coordinate). The slope of the line graphing this relation is

\[
\frac{212 - 32}{100 - 0} = \frac{9}{5}.
\]

This can be stated this way: a 9 degree change Fahrenheit is the same as a 5 degree change Celsius. Now we also know the y-intercept: it is 32, since (0,32) is on the graph. Thus the function relating Fahrenheit to Celsius is

\[
F = \frac{9}{5}C + 32.
\]

Now, express Celsius as a function of Fahrenheit.

2. This is an elaboration of the activity in Chapter 2, section 1. Select a cylindrical container large enough so as to contain half a gallon of water. Fill a small cup with water and pour it in into the cylinder. Make a mark, and weigh the cylinder. Add another cup of water, make a mark and reweigh the cylinder. Do this several times.
   a) Make a table of values using these variables: number of cups, height of the corresponding mark, weight of the water. On the same piece of graph paper, graph these data, with “cups” along the x-axis, and height and weight along the y-axis.
   b) Is height of the column of water a linear function of the volume of the column? Is the weight a linear function of the column of water? Are either of these a proportional relation?
   c) Your points probably don’t lie on a line, but are very close to it. Can you explain this?

3. A certain bank provides 4% interest at the end of each year on the amount of money in the account. Suppose that I open an account with $100, and just let the money collect in the account for 10 years. Write a table with “year” and “balance” as the variables, and plot these points. Do these data describe a linear relation?

4. Suppose you live in a community that has a swimming pool, and every day you record the air temperature at noon \(T\) and the number \(N\) of people in the pool. Do you expect \(N\) to be a linear function of \(T\)? Do you expect (or not expect) anything else about how \(N\) changes as \(T\) changes?

5. At Mario’s Cut Rate Used Car Lot, Mario compensates his salespeople with salary + commission; that is each salesperson receives a base salary and then a certain amount for each car sold. His more experienced people get a higher base salary, but the new people get a higher commission, because he want to encourage them to be eager to sell cars. Sally, his most seasoned salesperson receives a salary of $4,000 per month and a commission of $250 per car sold. Dmitri is a rookie, receives a salary of $2,800 per month, but his commission is $325 per car sold.
   a). Create a table of Sally’s and Dmitri’s earnings at 1,2,3,..., 10 cars sold. Using these data draw the graphs of Sally’s and Dmitri’s income as functions of \(N\): number sold.
   b) These lines intersect. Explain the significance of the point of intersection.
   c) Use the graph to solve this problem: Suppose Sally sells 14 cars in a month. How many cars does Dmitri have to sell to earn at least as much as Sally earns that month?
   c) Write down the formula for Sally’s earnings \(S\) as a function of number \(N\) of cars sold. Write down the formula for Dmitri’s earnings \(S\) as a function of number \(N\) of cars sold.
d) From the formula find the coordinates of the point of intersection.

6. The super express train from New York to Chicago travels at 215 mph.
   a) How many miles does it travel in $H$ hours? How many km in $H$ hours (remember 1 mile = 1.6 km). What is the rate in km/hour?.
   b) Chicago is 1100 kilometers from New York. How long does the trip take? Remember 1 mile = 1.6 km).

7. A bus traveling from Terminal A to Terminal B makes 8 stops. The bus starts out with 32 passengers. The following table shows how many people get on and off at each stop

<table>
<thead>
<tr>
<th>Stop 1</th>
<th>Stop 2</th>
<th>Stop 3</th>
<th>Stop 4</th>
<th>Stop 5</th>
<th>Stop 6</th>
<th>Stop 7</th>
<th>Stop 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>People Entering</td>
<td>4</td>
<td>7</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>11</td>
<td>2</td>
</tr>
<tr>
<td>People Exiting</td>
<td>9</td>
<td>3</td>
<td>6</td>
<td>0</td>
<td>12</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

How many people are on the bus when it leaves the last stop to return to Terminal A? Make a graph of “number $y$ of people on the bus” as a function of “bus approaching stop $x$.”

8. The state is building a road 4.5 km long from point A to point B. It takes the crew 3 weeks to complete 600 m. How long will it take them to complete the road? Draw a graph of km vs. weeks showing how long ($x$ weeks) it takes to complete a road of $y$ km.

Let us stop for a moment to point out a distinction among these functions. Those in examples 4, 5, 7 are such that the input and outputs are integers: there is no such thing as 3.4 people or 7.3 cars. Such relationships are described as *discrete*, in the sense that the inputs and output are not all numbers, but some selection of numbers. When the relationship is defined for all numbers, as that of °C and °F, we say that the relationship is *continuous*. This distinction is made in terms of context: clearly temperatures can take all possible values, but we cannot have buses destined for Price leaving Salt Lake City continuously.

9. A bookseller is trying to set a price for her books in such a way as to keep the carry-over inventory at an acceptable level. So, she decides to vary her prices, month by month for a little over a year, to see the relationship between price and inventory. Here are the data:

<table>
<thead>
<tr>
<th>Month</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>apr</th>
<th>May</th>
<th>Jun</th>
<th>Jul</th>
<th>Aug</th>
<th>Sep</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>1.4</td>
<td>1.10</td>
<td>1.00</td>
<td>1.4</td>
<td>1.80</td>
<td>2.40</td>
<td>1.60</td>
<td>2.00</td>
<td>2.50</td>
<td>3.50</td>
<td>2.65</td>
<td>1.50</td>
</tr>
<tr>
<td>Inventory</td>
<td>90</td>
<td>98</td>
<td>75</td>
<td>55</td>
<td>98</td>
<td>146</td>
<td>115</td>
<td>100</td>
<td>110</td>
<td>175</td>
<td>125</td>
<td>92</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Month</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>0.80</td>
<td>3.25</td>
<td>0.75</td>
</tr>
<tr>
<td>Inventory</td>
<td>90</td>
<td>160</td>
<td>70</td>
</tr>
</tbody>
</table>

We can’t tell much from these data, except that each time the price was lowered, the unsold inventory was lowered. Maybe, if we reorder according to price, with the inventory as the dependent variable, we get a different picture. In fact the picture we get is this:
We see several things that are not readily apparent from the table: generally speaking, as the price rises, so does the inventory. We could conclude that, if we never want an unsold inventory of more than 150 items, then we should keep the price under $2 (well, almost all the time, in one month out of 10 this was not true). We also don’t discern any curving of the data, so we might surmise that the relation (except for random variations) is linear.

In general, if we are given a table of data, we should first determine (from the context) which variable should be the horizontal, and which, the vertical. Then we should reorder the table in increasing order in \( x \). We can now check for linearity: if the change in \( y \) is proportional to the change in \( x \) (that is, given any two points, the quotient of these changes is always the same number), then the data are that of a linear relation. An easier way, and one which in any case gives good information, is to plot the points to see whether or not they lie on a line. The data may have come from measurements which are prone to random error. So, if the points almost lie on a line, but do not actually lie on a line, we may be able to conclude that the relation is linear.

**Analyzing a Functional Relationship**

Data that are collected from real contexts, such as a laboratory experiment, or a questionnaire, are very unlikely to fall on a line - or for that matter in any precise pattern. Nevertheless, the graphed data may show a trend or suggest a relation, or uncover an anomaly. As an example, let us look at the average hour by hour temperature for an August day in Salt Lake City:

<table>
<thead>
<tr>
<th>Time of day</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature</td>
<td>73</td>
<td>72</td>
<td>71</td>
<td>70</td>
<td>69</td>
<td>69</td>
<td>70</td>
<td>72</td>
<td>76</td>
<td>81</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time of day</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature</td>
<td>74</td>
<td>84</td>
<td>85</td>
<td>87</td>
<td>89</td>
<td>89</td>
<td>88</td>
<td>88</td>
<td>68</td>
<td>85</td>
<td>84</td>
</tr>
</tbody>
</table>

Plot the points on a graph, and connect those points with a smooth curve:
The graph confirms some things that we should have expected on physical grounds: that the temperature rises during the day as the sun moves directly overhead, and drops - more or less linearly - when the sun is down. We also see that the highest temperature is later in the day than we might have suspected; suggesting a cooling-off lag. Now let’s look at the data for a particular day: August 26, 2012:

<table>
<thead>
<tr>
<th>Time of day:</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature:</td>
<td>78</td>
<td>76</td>
<td>75</td>
<td>74</td>
<td>73</td>
<td>72</td>
<td>73</td>
<td>75</td>
<td>77</td>
<td>82</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time of day:</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature:</td>
<td>86</td>
<td>90</td>
<td>92</td>
<td>94</td>
<td>84</td>
<td>78</td>
<td>82</td>
<td>84</td>
<td>82</td>
<td>81</td>
<td>80</td>
</tr>
</tbody>
</table>

Here is the graph:

This starts out like a typical August day, but then there is a sudden decrease in temperature at 15:00 for about 2 hours. This suggests a thunderstorm or the arrival of a cold front. However, the data show
a warming up around 17:00, returning to the “typical day.” This is not what happens when a cold front arrives, so strengthens the argument for a thunderstorm.

Let’s look back at the previous graph of “average August temperatures.” Notice that the temperature rises at a steady state until about 3 pm where it flattens out a bit. Since this is the average temperature, this blip suggests that afternoon thunderstorms occur in August frequently enough to affect the average.

**Activities**

10. Given the following graphs create a context that could be described by the graph, and describe the features in terms of the context.

![Graph](figure12.png)

Figure 12

To illustrate what is intended in this activity, here is a possible context. There is a concert in the local stadium, and all seats are sold out. The graph shows the number of people in the stadium at time $t$, where the clock starts about 2 hours before the beginning of the event, and ends at about half an hour after the start of the event.

![Graph](figure13.png)

Figure 13
11. Rudolph starts a savings account. He puts $1000 in his account the first year, and each year adds $400. Komaiko started an account in the same year, but she put in $1, with the plan of adding a matching amount to the account in each year. How do they stand after 15 years?

12. Suppose you make a plot $H$, height, against $T$ age in years, from measurements made on some
person each year. Is $H$ a linear function of $T$? Draw a graph that you think represents the data that will have accrued in 21 years.

13. Draw a graph that you think represents the mean water depth (as a measure of the volume of water) in Lake Powell over a year, where the measurements are made monthly. Keep in mind the precipitation upstream and the use of water downstream.