Chapter 3. REPRESENTATIONS OF A LINE

In this chapter we shall study linear relations among quantities in detail in each of the realizations: formulas, tables, graphs and context, and develop fluidity in moving among them. Although the Utah Core Standards refers to the concept of function, in this chapter we continue this as a discussion of linear relations, which, when of the form $y = mx + b$ is called a function. The transition in thinking from equations to relations to functions is - as was that from “unknowns” to “variables” - profound. For this reason we have deferred that discussion to the next Chapter, after the study of linear relations is completed.

Section 3.1 Linear relationships in patterns and context

*Construct a function to model a linear relationship between two quantities. (8F4)*

Linear relations give rise to patterns. We also will see that certain kinds of patterns, called *additive*, give rise to linear relations.

**Example 1.** Consider the linear relation between the quantities $x$ and $y$: $5x + 3y = 84$. If we try $x = 0$, we get the equation $3y = 84$, or $y = 28$. Now rewriting the equation as $5x + 3y = 3(28)$, we see that, if $x$ is an integer, it must be divisible by 3, since the two other terms in the equation are divisible by 3. This is already an interesting pattern if we are looking for solutions that are integers: if two of the terms are divisible by a prime number, so is the third. So, in this case, let us introduce the new variable $z$ so that $x = 3z$. The equation becomes $5(3z) + 3y = 3(28)$, which simplifies to $5z + y = 28$. We can now make a nice table of integral values:

<table>
<thead>
<tr>
<th>$m$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>0</td>
<td>3</td>
<td>6</td>
<td>9</td>
<td>12</td>
<td>15</td>
</tr>
<tr>
<td>$y$</td>
<td>28</td>
<td>23</td>
<td>18</td>
<td>13</td>
<td>8</td>
<td>3</td>
</tr>
</tbody>
</table>

Note that every time $z$ increases by 1, $x$ increases by 3 and $y$ decreases by 5. In both cases, since the changes in $x$ and $y$ are constant multiples of the change in $z$, we know that the graphs of $x$ and $y$ against $z$ are straight lines.

The increasing line is that of $x$ against $z$, and the decreasing line is that of $y$ against $z$. Now, remember that $z$ was a quantity that we chose to introduce, and not part of the given equation. So, finally we display the graph of $y$ against $x$.
Example 2. Suppose we start with a table of pairs of numbers, and we want to know if they are related by a linear relation, and if so, how to find the relation. Consider

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>8</td>
<td>11</td>
<td>14</td>
<td>17</td>
<td>20</td>
<td>23</td>
</tr>
</tbody>
</table>

Each unit change in \(x\) corresponds to a 3 unit change in \(y\), so the unit rate of change is 3, and this is a linear relation.

For any table where rate of change of one variable with respect to the other is not constant, the graph of these data will not be a straight line.

Example 3. Consider the diagram below:

There is a pattern here: each time we move to the right by one unit, the height of the stack increases by 2. If we let \(x\) be the horizontal axis, and \(y\) the vertical, we see that \(y\) starts out as 3 and then, at each move of \(x\) to the right, increases by 2. So, \(y\), for \(x\) moves to the right, is equal to \(3 + 2x\). This process gives us the following graph:

Example 4. Typically a car salesperson has a base monthly salary, supplemented by a commission on each sale. This gives us a pattern of steady growth: with each additional car sold, the salesperson earns an additional amount: the commission. To illustrate, suppose that the base monthly salary is $2000 and the commission for each car sold is $300. We can illustrate this by a table:

<table>
<thead>
<tr>
<th>Number of cars sold</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compensation</td>
<td>2000</td>
<td>2300</td>
<td>2600</td>
<td>2900</td>
<td>3200</td>
<td>3500</td>
</tr>
</tbody>
</table>

and also describe this situation with a graph:

Section 3.2. Graphing and Writing Equations of Lines

Interpret the equation \(y = mx + b\) as defining a linear function whose graph is a straight line. (8F.3)

Determine the rate of change and initial value of the function form a description of a relationship, or from two \((x, y)\) values, including reading from a table or a graph. (8F.4)
In the last example of Chapter 2, we looked at the line through the two points \((0,5)\) and \((2,9)\), and calculated the rise/run to find that the slope of the line is 2. Now, since the slope can be calculated using any two lines, we can create this test for the the point \((x, y)\) to be on the line: the slope calculation using this point and either of the points \((0,5)\) or \((2,9)\) must give the value 2:

\[
\frac{y - 9}{x - 2} = 2
\] (1)
Such a test is called the equation of the line: not only does this equation hold when \((x, y)\) is on the line; if this calculation gives a number other than 2, \((x, y)\) cannot be on the line, for it lies on the line through \((2,9)\) with that different slope. So, we ask: is \((3,10)\) on the line? We calculate the slope of the line segment between \((3,10)\) and \((2,9)\), and get 1. Thus \((3,10)\) is not on the line. But \((3,11)\) is a point on the line, since \((11-9)/(3-2) = 2\).

The (1) equation then gives a condition for \((x, y)\) to be on the line: it is an explicit relation that \(x\) and \(y\) must satisfy. With a little algebra, we simplify it: first multiply through by \(x-2\) to get as the condition:

\[ y - 9 = 2(x - 2) , \quad \text{or} \quad y - 9 = 2x - 4 \]

which simplifies even further to \(y = 2x + 5\).

Suppose that we do this calculation for the point \((x, y)\) and some other point on the line; say \((-1,3)\). For \((x, y)\) to be on the line we must have

\[ \frac{y - 3}{x - (-1)} = 2 \]

or \(y - 3 = 2(x + 1)\), which also simplifies to \(y = 2x + 5\).

To summarize, once we know the slope of the line, we can use any point on the line to calculate the condition a general point \((x, y)\) has to satisfy to be on the line. And if we know two points on the line, we can use those points to give us the slope. Finally, the equation for the computation of the slope given the generic point \((x, y)\) and a point on the line, can always be written in the form \(y = mx + b\), called the point-slope form of the equation of a line because \(m\) is the slope, and \((0,b)\) the y intercept is on the line. No matter what points on the line we choose for the calculations, the point-slope form of the equation will always be the same.

**Example 5.** Given the point \(P:(3,5)\) and the number \(m = -1\), find the equation of the line through \(P\) with slope \(m\).

**Solution.** Following the above, the point \(X : (x,y)\) is on the line if the slope calculation with the points \(X\) and \(P\) gives -1:

\[ \frac{y - 5}{x - (3)} = -1 . \]
This simplifies to \( y - 5 = -(1)(x - 3) \), or \( y - 5 = -x + 3 \), and finally \( y = -x + 8 \). **Example 6.** Given the points \((2,1), (-1,10)\), find the equation of the line through those points. First we calculate the slope using the given points:

\[
\frac{10 - 1}{-1 - 2} = \frac{9}{-3} = -3
\]

Now, the equation of the line is given by the slope calculation using the generic point \((x, y)\) and one of the given points (say \((2,1)\)):

\[
\frac{y - 1}{x - 2} = -3
\]

or, \( y - 1 = -(3)(x - 2) \), which simplifies to \( y = -3x + 7 \).

**Example 7.** Given a straight line on the coordinate plane, find its equation. Let’s do this with the following graph

![Figure 6](image)

We look for convenient pairs of points on the line to calculate the slope: \((1.0, 2.8)\) and \((2,4)\) look good, so let’s choose them. Calculate, or simply observe, that as \(x\) increases by 1, \(y\) increases by 1.2, so the slope is \( m = 1.2 \). Now I write the slope calculation using the points \((x, y)\) and \((2, 4)\)

\[
\frac{y - 4}{x - 2} = 1.2
\]

or \( y - 4 = (1.2)(x - 2) \) which gives the point-slope form \( y = (1.2)x + 1.6 \). Note: one must always pay attention to the scale on a diagram; in the case of figure 6, the scale is different on each axis. On the \(x\) axis the increments are 0.2, and on the \(y\), they are 0.4.

*Compare properties of functions (linear) presented in a different way (algebraically, graphically, numerically in tables or by verbal descriptions 8F2).*

*Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or table of values (8F4).*

Four ways to study linear relations are: the equation, the graph, a table, and context. We have touched on these in the above sections, and the students have seen these from time to time over the past few years. Here we summarize this circle of ideas.
**Example 8.** The sum of two numbers is 5. How many pairs of numbers can you list that will satisfy this condition?

Use $x$ and $y$ to represent the two numbers. The sum of $x$ and $y$ is $x + y$; the assertion is that this is 5. This statement can be expressed by the equation

$$x + y = 5.$$ 

Now, we can subtract $x$ from both sides of $x + y = 5$, giving us $y = 5 - x$, giving us an explicit rule for finding $y$ given $x$: subtract $x$ from 5. If we write this as $y = -x + 5$, we have the point-slope form of the line.

Typically, the first thing to do in trying to understand a relation is to make a table of solutions to see what information we can gather. Then, plot the data points on a graph, and connect them. So for $x + y = 5$ we make the following table.

<table>
<thead>
<tr>
<th>x</th>
<th>-8</th>
<th>0</th>
<th>1</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>13</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>-3</td>
</tr>
</tbody>
</table>

From the table we see that as $x$ increases, $y$ decreases. In fact, whenever $x$ increases by 1, $y$ decreases by 1, confirming that the slope is -1.

Here is the graph:

![Graph](image)

Figure 7

Connecting the points on the graph, we get a straight line. Every point on the graph is a solution, even if it wasn’t included in our list of solutions. For any point $(x, y)$ on the graph, if we add $x$ and $y$ we get 5.

Notice that if we shift the line down by 5 units (or to the left by 5 units) we get the graph of the relation $x + y = 0$. This is because a shift downward by 5 units subtracts 5 from the $y$ coordinate, and a shift to the left by 5 units subtracts 5 from the $x$ coordinates; in either case $x + y$ is reduced by five units, and so we must have $x + y = 0$ for the new line.

**Example 9.** a) Translate the line $y = x$ upwards by two units, so that the point $(x, x)$ goes to $(x, x + 2)$. In particular, notice the equation of the new line is $y = x + 2$. 

6
b) Show that the graph of \( y = 2x + 2 \) coincides with the shift upwards by 2 units of the graph of \( y = 2x \).

**Example 10.** Graph the relation \( y = 2x - 5 \). First construct a table by choosing “typical” numbers for \( x \):

<table>
<thead>
<tr>
<th>x</th>
<th>-4</th>
<th>-2</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>-13</td>
<td>-9</td>
<td>-5</td>
<td>-1</td>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

Note that every increase in \( x \) by 2 is accompanied by an increase in \( y \) by 4. Here is the graph:

![Figure 8](image)

If we shift this graph upwards by 5 units, it will pass through the origin and now be the graph of the proportional relation \( y = 2x \).

To summarize: to find the equation of the line through the two points \((x_0, y_0), (x_1, y_1)\), first calculate the slope of the line: points on the line, then

\[
m = \frac{y_1 - y_0}{x_1 - x_0}.
\]

If \((x, y)\) is any point in the plane then it is on the line \(L\) if and only if the slope calculation gives \( m \):

\[
\frac{y - y_0}{x - x_0} = m.
\]

If the variables \( x, y \) are in a linear relation (that is, the graph of the relation is a line), the relation can be expressed in the form of the function \( y = mx + b \), where \( m \) is the slope and \( b \) is the \( y \)-intercept.

Now, suppose that \( x \) and \( y \) are related by the equation \( ax + by = c \), where \( a, b, c \) are specific numbers. We assume that one of \( a, b \) is not zero, otherwise we do not have a relation, just the statement \( c = 0 \). If \( b = 0 \), then the equation of the line can be written as \( x = c/a \) (the case of a vertical line). If \( b \neq 0 \), we can divide by \( b \), and move the \( x \) term to the other side to get the function \( y = -(a/b)x + c/b \) representation of the line.

**Section 3.3. Parallel and Perpendicular lines**
Example 11. Let us return to the water measurement activity at the beginning of chapter 2, and plot both sets of weight data as the $y$ coordinate, and the height on the $x$-axis. Connecting the two sets of data with a line, we get the following graph, where blue is the measured weight and green is the weight of the water.

In all of the examples discussed above, we talked about changes in the relation between the two variables that results from a shift, or a translation. To make this precise: a translation by $(a, b)$ is a motion of the plane by $a$ horizontally, followed by a motion of the plane by $b$ vertically.

Example 12. Consider the graph of $x + y = 5$, as described in example 81. Shift the graph upward by 3 (that is by $(0, 3)$): what is the equation of the new line?

Solution. Since we have increased $y$ by 3, $x + y$ has increased by 3, so the equation of the new line $x + y = 8$. Here is another way to see this. Let (new $x$, new $y$) be the coordinates of the point to which (old $x$, old $y$) is moved. We know that

$$\text{old } x = \text{new } x, \quad \text{old } y = \text{new } y - 3.$$

The equation of the old line is

$$\text{old } x + \text{old } y = 5$$

which is, in terms of the new coordinates:

$$\text{new } x + \text{new } y - 3 = 5.$$

Since we are drawing the lines on the same coordinate plane, we can remove the word “new” to get

$$x + y - 3 = 5, \quad \text{or} \quad x + y = 8.$$ 

Finally we can verify these observations with the table:

<table>
<thead>
<tr>
<th>x:</th>
<th>-8</th>
<th>0</th>
<th>1</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Old y:</td>
<td>13</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>-3</td>
</tr>
<tr>
<td>New y:</td>
<td>16</td>
<td>8</td>
<td>7</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>
Example 13. Translate the line \( y = 2x \) by one unit in each coordinate, so that \((x,2x)\) goes to \((x+1,2x+1)\). Find the equation of the new line.

Example 14. Show that, if we translate the graph of \( x + y = 5 \) by \((a,b)\), where \(a+b=3\), the line goes to the graph of \( x + y = 8 \).

<table>
<thead>
<tr>
<th>If the line ( y = mx ) is translated by ((a,b)), then the equation of its image is</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y - b = m(x - a) )</td>
</tr>
</tbody>
</table>

Now, two lines are said to be parallel if there is a translation that takes one into the other. The statement in the box tells us that if two lines are parallel, they have the same slope. It is also true that if two lines have the same slope, they are parallel, that is: there is a translation that takes one into the other. How do we find that translation? Consider this figure of two lines with the same slope:

Each orange arrow represents a translation. Observe, using two pieces of transparent graph paper, that each of those translations takes line 1 to line 2. You should conclude that, given any points \( P \) on line 1, and \( Q \) on line 2, the translation from \( P \) to \( Q \) takes line 1 to line 2. Since this diagram can be of any two lines with the same slope (we have omitted coordinates to emphasize this point); we can conclude

| Two lines are parallel if and only if there is a translation of one line to the other. Parallel lines have the same slope and lines with the same slope are parallel. |
In chapter 8 where we will study translations in more detail, we will note that if two lines are parallel they never intersect, and conversely, if two lines never intersect they are parallel. This statement (a version of the Parallel Postulate of Euclid) cannot be verified by observation because we cannot see infinitely far away. For this reason, it has been discussed throughout history, the issue being whether or not it is a necessary part of planar geometry. It turned out, in the 19th century, that it is, for there are geometries different from planar that satisfy all of the conditions of planar geometry but the Parallel Postulate.

**Activities.**

1. The two lines in figure 10 are parallel. By picking points on the lines, show that the lines have the same slope. Find the equations of the two lines. Explain, in terms of the equations of the lines, how to find a translation of one line to the other.

![Figure 10](image)

2. Here is a graph of two lines

![Graph of two lines](image)

a) Find equations of the two lines.
b) By how much does the blue line have to be shifted vertically to land on top of the red line? Describe the shift as a translation.

c) By how much does the blue line have to be shifted horizontally to land on top of the red line? Describe the shift as a translation.

d) Find numbers \(a\) and \(b\) for which, when we substitute \(x + a\) for \(x\) and \(y + b\) for \(y\) in the equation of the first line, we get the equation of the second line.

3). Graph the two lines given by these equations:

\[
L : \quad 2x + y = 7 \quad 4x + 2y = 9.
\]

Explain, both in terms of the graphs (see figure 11) and in terms of the equations, why the two lines are parallel.

![Figure 11](image)

4. \(3x - 2y = 1\) and \(3x - 2y = 7\) are parallel lines. Find a substitution \(x + a\) for \(x\) and \(y + b\) for \(y\) which produces the equation of the second line.

5. \(3x - 2y = 1\) and \(6x - 4y = 38\) are parallel lines. Find a substitution \(x + a\) for \(x\) and \(y + b\) for \(y\) which produces the equation of the second line.

Two lines are **perpendicular** if they intersect, and all angles formed at the intersection are equal. This is of course is the same as saying that all these angles have measure 90°.

To understand perpendicularity, we will need the idea of **rotation**. A rotation is a motion of the plane around a point, called the center of the rotation. To visualize what a rotation is, take two pieces of transparent coordinate paper, put one on top of the other and stick a pin through both piece of paper. The point where the pin intersects the paper is the center of the rotation. Now any motion of the top piece of paper is a visualization of a rotation: For any figure on the bottom piece of paper, copy it onto the top, then rotate the top piece of paper and copy the figure on the top to the bottom. That image is the rotated image of the original figure.

In figure 12, we see the result of rotating the red line through a right angle (90°) with the center \(C\): the blue line is the image of the red line under the rotation.
Notice that the dark lines and the gray lines correspond under the rotation, so they have the same lengths. Notice also that these are the triangles that are drawn for the slope computation except that the rise and run has been interchanged: in terms of lengths, \( \text{rise(red)} = \text{run(blue)} \), \( \text{run(red)} = \text{rise(blue)} \). However, there is one serious omission so far: the slope computation is in terms of differences between coordinates, and not lengths. In our diagram the sign of one pair of differences (represented by the black lines) has changed, while the sign of the other pair of differences. We can summarize this as follows:

For the red line,

\[
\text{slope(red)} = \frac{\text{length(black)}}{\text{length(gray)}},
\]

and for the blue line

\[
\text{slope(blue)} = \frac{\text{length(gray)}}{\text{length(black)}},
\]

from which we can conclude that the product of the slopes of the blue and red lines is -1. Since we did not use any coordinates to make this argument, this statement is general, so long as neither line is horizontal or vertical. In fact, we can conclude more: if the product of the slopes of two lines is -1, then they are perpendicular at their point of intersection. For a rotation of one line by 90° takes it onto a line of the same slope of the other, and so must be the other, since a line is determined by a point and a slope. Thus

\[
\text{If lines } L_1 \text{ and } L_2 \text{ are perpendicular at their point of intersection, then the product of their slopes is -1. If the product of the slopes of lines } L_1 \text{ and } L_2 \text{ is -1, then they are perpendicular at their point of intersection.}
\]

**Activity**

a) Consider the line \( L \) given by the equation \( 3x + 4y = 20 \) Draw its graph. Note that the point \( P(2.4, 3.2) \) is on the line. Rotate \( L \) by 90° about \( P \). \( L \) has moved to a line \( L' \) perpendicular to \( L \). Find the equation of \( L' \). Calculate the slopes of \( L \) and \( L' \).

b) Do the same activity with another line of your choice, and calculate the results

We observe from these calculations that in all cases, the rise and run interchange, and there is a sign change in just one of them.