Chapter 10. Angles, Triangles and Distance

In section 1 we begin by gathering together facts about angles and triangles that have already been discussed in previous grades. This time the idea is to base student understanding of these facts on the transformational geometry introduced in the preceding chapter. As before, here the objective is to give students an informal and intuitive understanding of these facts about angles and triangles; all this material will be resumed in Secondary Mathematics in a more formal and logically consistent exposition.

Section 2 is about standard 8G6:

*Explain a proof of the Pythagorean Theorem and its converse.*

The language of this standard is very precise: it does not say *Prove...* but it says *Explain a proof of...*, suggesting that the point for students is to articulate their understanding of the theorem; not to demonstrate skill in reciting a formal proof. Although Mathematics tends to be quite rigorous in the construction of formal proofs, we know through experience, that informal, intuitive understanding of the "why" of a proof always precedes its articulation. Starting with this point of view, the student is guided through approaches to the Pythagorean theorem that make it believable, instead of formal arguments. In turn, the student should be better able to explain the reasoning behind the Pythagorean theorem, than to provide it in a form that exhibits form over grasp.

In Chapter 7, in the study of tilted squares, this text suggests that by replacing specific numbers by generic ones, we get the Pythagorean theorem. We start this section by turning this suggestion into an “explanation of a proof,” and we give one other way of seeing that this is true. There are many; see, for example

jwilson.coe.uga.edu/EMT668/emt668.student.folders/HeadAngela/essay1/Pythagorean.html

The converse of the Pythagorean theorem states this: if \(a^2 + b^2 = c^2\), where \(a, b, c\) are the lengths of the sides of a triangle, then the triangle is right, and the right angle is that opposite the side with length \(c\). The Euclidean proof of this statement is an application of SSS for triangles. Although students played with SSS in Chapter 9, here we want a more intuitive and dynamic understanding. We look at the collection of triangles with two side lengths \(a\) and \(b\) with \(a \geq b\). As the angle at \(C\) grows from very tiny to very near a straight angle, the length of its opposing side steadily increases. It starts out very near \(a - b\), and ends up very near \(a + b\). There is only on triangle in this sequence where \(a^2 = b^2\) is precisely \(c^2\).

In the final section, we use the Pythagorean Theorem to calculate distances between points in a coordinate plane. This is what the relevant standard asks: it does not ask that students *know* the “distance formula.” The goal here is that students understand the **process** to calculate distances: this process involves right angles and the Pythagorean theorem and students are to understand that involvement. Concentration on the formula perverts this objective.
Section 10.1 Angles and Triangles.

Use informal arguments to understand basic facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. 8G5

In this section, we continue the theme of the preceding chapter: to achieve understanding through exploration. We start with geometric facts that students learned in 7th grade or earlier, exploring them from the point of view of rigid motions and dilations.

(1). Vertical angles at the point of intersection of two lines have the same measure. The meaning of vertical is in the sense of a vertex. Thus the angles at V with arrows are vertical angles, as is the pair at V without arrows.

Now rotation through a straight angle (180°) takes a line into itself, but changes the direction. So rotation by a straight angle with center V takes segment VA to VC and VB toVD, and so carries $\angle AVB$ to $\angle CVD$, so the two angles have the same measure.

We are motivated to argue on the basis of rigid motions when we can. The traditional argument is this: both angles $\angle AVB$ and $\angle CVD$ are supplementary to $\angle BVC$ (recall that two angles are supplementary if they add to a straight angle), and therefore must have equal measure.

(2). If if two lines are parallel, and a third line L cuts across both, then corresponding angles at the points of intersection have the same measure.
In figure 2, the two parallel lines are $L'$ and $L''$, and the corresponding angles are as marked at $P'$ and $P''$. The translation that takes the point $P'$ to the point $P''$ takes the line $L'$ to the line $L''$ because a translation takes a line to another one parallel to it, and by the hypothesis, $L''$ is the line through $P''$ parallel to $L'$. Since translations also preserve the measure of angles, the corresponding angles as marked (at $P'$ and $P''$) have the same measure. Now, because of proposition one, that opposing angles at a vertex are of equal measure, we can conclude that $\angle A'P'P''$ and $\angle P''P'C''$ also have equal measure.

This figure also demonstrates the converse statement:

(3). Given two lines, if a third line $L$ cuts across both so that corresponding angles are equal, then the two lines are parallel.

To show this, we again draw figure 2, but now the hypothesis is that the marked angles at $P'$ and $P''$ have the same measure. Since translations preserve the measure of angles, the translation of $P'$ to $P''$ takes the angle $\angle A'P'P''$ to $\angle B''P''A''$, and so the image of $L'$ has to contain the ray $P''B''$, and so is the line $L''$. Since the line $L''$ is the image of $L'$ under a translation, these lines are parallel.

(4). The sum of the interior angles of a triangle is a straight angle.

In seventh grade, students saw this to be true by drawing an arbitrary triangle, cutting out the angles at the vertices, and putting them at the same vertex. Every replication of this experiment produces a straight angle. So, the evidence is overwhelming: it has to be true, but the evidence does not tell us why.

We have two arguments to show why it is true. The first has the advantage that it uses a construction with which the student is familiar (that to find the area of a triangle) and thus reinforces that idea. The second has the advantage that it can be generalized to polygons with more sides. First, draw a triangle with a horizontal base, as in figure 3. Rotate a copy of the triangle around the vertex $B$, and then translate the new triangle upwards to get the result shown in figure 3, where the triangle with dashed sides is the new position of the copied triangle. We have indicated the corresponding angles with the greek letters $\alpha, \beta, \gamma$. Since the angles $\angle ABC$ and $\angle C'B'A'$ have the same measure ($\beta$), the lines $AB$ and $B'A'$ are parallel. Since they are parallel, the angles $B'A'C'$ and $\angle A'C'E$ have the same measure.

Now look at the point $B = C'$: the angles $\alpha, \beta, \gamma$. form a straight angle.

An alternative argument is based on figure 4 below: In this figure we have named the “exterior angles” of the triangle, $\alpha', \beta', \gamma'$, each of which is outside of the triangle formed by the extension of the side of the triangle on the right. If we were to walk around the perimeter of the triangle, starting and ending at $A$ looking in the direction of $A'$, we would rotate our line of vision by a full circle, $360^\circ$. As this is the sum of the exterior angles, so we have

$$\alpha' + \beta' + \gamma' = 360^\circ.$$

But each angle in this expression is supplementary to the corresponding angle of the triangle,
that is, the sum of the measures of the angle is $180^\circ$. So, the above equation becomes

$$(180^\circ - \alpha) + (180^\circ - \beta) + (180^\circ - \gamma) = 360^\circ,$$

from which we get $\alpha + \beta + \gamma = 180^\circ$.

We can generalize the second argument to polygons with more sides. Consider the quadrilateral in figure 5.

By the same reasoning as in figure 4 (for the triangle), the sum of the exterior angles of the quadrilateral is also $360^\circ$ and the sum of each interior angle and its exterior angle is $180^\circ$. But now there are four angles, so we end up with the equation

$$(180^\circ - \alpha) + (180^\circ - \beta) + (180^\circ - \gamma) + (180^\circ - \delta) = 360^\circ,$$

or $720^\circ - (\alpha + \beta + \gamma + \delta) = 360^\circ$.

(5). The sum of the interior angles of a quadrilateral is $360^\circ$.

Can you now show that, for a five sided polygon, the sum of the interior angles of a quadrilateral is $540^\circ$? Can you go from there to the formula for a general polygon?
(6). If two triangles are similar, then the ratios of the lengths of corresponding sides is the same, and corresponding angles have the same measure.

(7). Given two triangles, if we can label the vertices so that corresponding angles have the same measure, then the triangles are similar.

We saw in Chapter 9 why (6) is true. Let us look more closely at statement (7).

Figure 6a shows a possible configuration of the two triangles. By a translation, we can place point $A$ on top of point $A'$ to get Figure 6b. Now move the smaller triangle by a rotation with center $A = A'$, so that the point $C$ lands on the segment $A'C'$. Since the angles $\angle CAB$ and $\angle C'A'B'$ have the same measure, the rotation must move line segment $AB$ so that it lies on $A'B'$. Now, since $\angle ACB$ and $\angle A'C'B'$ have the same measure, the line segments $CB$ and $C'B'$ must be parallel (by Proposition 2). Now the dilation with center $A$ that puts point $C$ on $C'$, puts triangle $ABC$ onto triangle $A'B'C'$, so they are similar.

The argument is not fully completed, for the configuration of figure 6a is not the only possibility. Consider the configuration of figure 7: now how do you find the desired similarity transformation? Again, we translate so that the points $A$ and $A'$ coincide. But now, the rotation that puts the line segment $AB$ on the same line as $A'B'$ doesn’t lead to the configuration of figure 6b and the smaller triangle cannot be rotated so that corresponding sides lie on the same ray. But this is fixed by reflecting the smaller triangle in the line containing the segment $AB$ and $A'B'$, and now we are in the configuration of figure 6b. Indeed, we could have started with a reflection of the small triangle in the line through $AC$, and then followed the original argument.

Have we covered all cases? The answer is yes: the difference between figure 6a and that of figure 7 is that of orientation. So, if we start again with the two triangles $ABC$ and
with corresponding angles of the same measure, then we should first ask: is the orientation $A \rightarrow B \rightarrow C$ the same as the orientation $A' \rightarrow B' \rightarrow C'$ (both clockwise or both counter clockwise)? If so, we are in the case of figure 6a. If not, after reflection in any side of triangle $ABC$ puts us in the case of figure 7.

Section 10.2 The Pythagorean Theorem.

*Explain a proof of the Pythagorean Theorem and its converse.* 8G6

In Chapter 7 we constructed “tilted” squares whose area was a specific integer, denoted $c^2$, but whose side had a length, $c$, that could not be expressed as a rational number (a quotient of integers). At the end of the discussion we mentioned that these specific examples were illustrations of a general theorem, known as the Pythagorean theorem. This mathematical fact is named after a sixth century BCE mathematical society (presumed to be led by someone named *Pythagoras*). It is clear that this was known to much earlier civilizations: the written record shows it being used by the Egyptians for land measurements, and an
ancient Chinese document even illustrates a proof. But the Pythagorean Society was given the credit for this by third century BCE Greek mathematicians. The Pythagorean Society also discovered the existence of line segments whose length cannot be represented by a quotient of integers (that is, irrational numbers), using this theorem, as we have discussed in Chapter 7. The legend is that the discoverer of this fact was sacrificed by the Pythagoreans. We mention this only to highlight how much the approach to mathematics has changed in 2500 years; in particular this fact, considered “unfortunate” then, is now appreciated as a cornerstone of the attempt to fully understand the concept of number, and its relationship to geometry.

Let’s pick up with the discussion at the end of Chapter 7, section 1. There we placed our tilted squares in a coordinate plane so as to be able to more easily see the relationship between the areas of the squares and its associated triangles. Here, in order to stress that the understanding of the Pythagorean theorem does not involved coordinates, we look at those Chapter 7 arguments in a coordinate free plane. For two positive numbers, $a$ and $b$, construct the square of side length $a + b$. This is the square bounded by solid lines in figure 8, with the division points between the lengths $a$ and $b$ marked on each side. Draw the figure joining these points - this is the square with dashed sides in figure 8. The original square consists of this square and four congruent triangles of leg lengths $a$ and $b$. This allowed us, in Chapter 7 to calculate the area of the tilted square, since the area of the triangles and of the full square are easily calculated from $a$ and $b$. Now, the hypotenuse of the triangle, whose length is denoted by $c$, is the side of the tilted square (with the dashed edges), so we have given an example of a number ($c^2$) that is easily calculable, but whose square root could very well not be.

Now, move to figure 9 which is the same original square of side length $a + b$, but is subdivided in a different configuration: the bottom left corner is filled with a square of side length $b$, and the upper right corner, by a square of side length $a$. The rest of the big square of figure
Figure 9 is a pair of congruent rectangles. By drawing in the diagonals of those rectangles as shown, we see that this divides the two rectangles into four triangles, all congruent to each of the four triangles of figure 8. Thus what is outside those rectangles must have area $c^2$. On the other hand (as we see in figure 9), what is outside those rectangles consists of a square of side length $a$ and another of side length $b$, so has area $a^2 + b^2$. This result is:

**The Pythagorean Theorem:** $a^2 + b^2 = c^2$ for a right triangle whose leg lengths are $a$ and $b$ and whose hypotenuse is of length $c$.

Given the statement of the standard, we are compelled to ask whether or not the comparison of the two images above is a proof. The answer is easy: it is not. The contemporary cinema demonstrates for us that pictures can be very convincing while portraying something quite removed from reality. Is this not the case in the comparison of figures 8 and 9. Well, it is not, and it is the rigorous proof that tells us that it is not: this proof involves an algebraic computation relating the formulas for the areas of those triangles and squares. But the proof adds little to understanding the pythagorean theorem provided by the figures above. The intent of this standard is to have the student understand the comparison of figures 8 or 9, or any one of the proofs at the website http://www.cut-the-knot.org/pythagoras/index.shtml.

**Example 1.**

- a) A right triangle has leg lengths 6 in and 8 in. What is the length of its hypotenuse?
  
  Let $c$ be the length of the hypotenuse. By the Pythagorean theorem, we know that
  
  $$c^2 = 6^2 + 8^2 = 36 + 64 = 100,$$
  
  so $c = 10$. 

b) Another triangle has leg lengths 20 in and 25 in. Give an approximate value for the length of its hypotenuse.

\[ c^2 = 20^2 + 25^2 = 400 + 625 = 1025 = 25 \times 41. \]

So, \( c = \sqrt{25 \times 41} = 5 \times \sqrt{41}. \) Since 6\(^2\) = 36 and 7\(^2\) = 49, we know that \( \sqrt{41} \) is between 6 and 7; probably a bit closer to 6. We calculate\(^7 \) 6.4\(^2\) = 40.96, so it makes good sense to use the value 6.4 to approximate \( \sqrt{41} \). Then the corresponding approximate value of \( c \) is 5 \times 6.4 = 32.

Example 2.

a) The hypotenuse of a triangle is 25 ft, and one leg is 10 ft long. How long is the other leg?

Let \( b \) be the length of the other leg. We know that \( 10^2 + b^2 = 25^2 \), or \( b^2 + 100 = 625 \), and thus \( b^2 = 525 \). We can then write \( b = \sqrt{525} \). If we want to approximate that, we first factor 525 = 25 \times 21, sp \( b \approx 5\sqrt{21} \). We can approximate \( \sqrt{21} \) by 4.5 (4.5\(^2\) = 20.25), and thus \( b \) is approximately given by 5 \times 4.5 = 22.5.

b) An isosceles right triangle has a hypotenuse of length 100 cm. What is the leg length of the triangle.

Referring to to the Pythagorean theorem, we are given: \( a = b \) (the triangle is isosceles), and \( c = 100 \). So, we have to solve the equation \( 2a^2 = 100^2 \). Since 100\(^2\) = 10\(^4\), we have to solve \( 2a^2 = 10^4 \), or \( a^2 = 5 \times 10^3 \). Write \( 5 \times 10^3 = 50 \times 10^2 \), which brings us to \( a = \sqrt{50 \times 10^2} = 10 \times \sqrt{50} \). Since 7\(^2\) = 49, we can give the approximate answers \( a = 10 \times 7 = 70 \).

Example 3. Figure 10 is that of an isosceles right triangle, \( \triangle ABC \), lying on top of a square. The total area of the figure is 1250 sq. ft. What is the altitude (\( CD \)) of the triangle?

Since \( CD \) is the altitude of the right triangle, it is perpendicular to the base \( AD \). Since the triangle is isosceles, the measure of \( \angle CAD \) and \( \angle CBD \) are both 45\(^\circ\). That tells us that triangles \( \triangle ADC \) and \( \triangle BDC \) are also isosceles right triangles. Let \( h \) be the length of the altitude, \( CD \). Then both \( AD \) and \( BD \) have length \( h \) as well, and the square beneath the triangle has side length 2\( h \), so has area \( 4h^2 \). The area of the triangle on top of the square (one-half base times altitude) is \( \frac{1}{2} (2h)(h) = h^2 \). So, the area of the entire figure is \( 4h^2 + h^2 = 5h^2 \), and we are given that that is 1250 sq. ft. We then have

\[ 5h^2 = 1250 \quad \text{so} \quad h^2 = 250 = 25 \times 10 \quad \text{and} \quad h = 5\sqrt{10}. \]

Converse of the Pythagorean Theorem. For a triangle with side lengths \( a, b, c \) if \( a^2 + b^2 = c^2 \), then \( \angle ACB \) is a right angle.
In order to see why this is true, let us show how to draw all triangles with two side lengths \(a\) and \(b\). Let’s suppose that \(a \geq b\). On a horizontal line, draw a line segment \(BC\) of length \(a\). Now draw the semicircle whose center is \(C\) and whose radius is \(b\). See figure 11. Then, any triangle with two side lengths \(a\) and \(b\) is congruent to a triangle with one side \(BC\), and the other side the line segment from \(C\) to a point \(A\) on the circle. Now, as the line segment \(AC\) is rotated around the point \(C\), the length of the line continually increases. When \(AC\) is vertical, we have the right triangle, for which the length \(c = \sqrt{a^2 + b^2}\). We conclude that for any triangle with side lengths \(a\) and \(b\), the length \(c\) of the third side is either less than \(\sqrt{a^2 + b^2}\) (triangle is acute), or greater than \(\sqrt{a^2 + b^2}\) (triangle is obtuse).
Example 4. Draw a circle and its horizontal diameter (AB in the accompanying figure). Pick a point C on the circle. Verify by measurement that triangle ABC is a right triangle.

For this particular triangle, the measures of the side lengths, up to nearest millimeter are: \(AB = 44\) mm, \(BC = 18\) mm, \(AC = 40\) mm. Now, calculate: \(BC^2 + AC^2 = 324 + 1600 = 1924\), and \(AB^2 = 1936\). This is pretty close If all students in the class get this close, all with different figures, then that is substantial statistical evidence for the claim that the triangle is always a right triangle.

[[In the honors sections they can try to show this in general. Let the circle be the unit circle centered at the origin, so \(A\) has coordinates \((-1,0)\) and \(B\) has coordinates \((1,0)\). Let the coordinates of \(C\) be \((x,y)\). Then the slope of \(AC\) is \(y/(x+1)\) and the slope of \(BC\) is \(y/(x-1)\). Let’s calculate the product of the slopes:

\[
\frac{y}{x+1} \cdot \frac{y}{x-1} = \frac{y^2}{x^2 - 1}.
\]

But \(C\) is on the unit circle, so \(x^2 + y^2 = 1\), and thus \(y^2 = 1 - x^2\), and the product of the slopes is -1.]]
Section 10.3 Applications of the Pythagorean Theorem.

Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions. 8G7

Example 5. What is the length of the diagonal of a rectangle of side lengths 1 inch and 4 inches?

The diagonal is the hypotenuse of a right triangle of side lengths 1 and 4, so is of length \( \sqrt{1^2 + 4^2} = \sqrt{17} \).

Example 6. Suppose we double the lengths of the legs of a right triangle. By what factor does the length of the diagonal change, and by what factor does the area change?

This situation is illustrated in figure 12, where the triangles have been moved by rigid motions so that they have legs that are horizontal and vertical, and they have the vertex A in common. But now we can see that the dilation with center A that moves B to B' puts the smaller triangle on top of the larger one. The factor of this dilation is 2. Thus all length change by the factor 2, and area changes by the factor \( 2^2 = 4 \).

Example 7. A room is in the shape of a rectangle of width 12 feet, length 20 feet, and height 8 feet. What is the distance from one corner of the floor (point A in the figure) to the opposite corner on the ceiling?

In the figure below, we want to find the length of the dashed line from A to B. Now, the dash-dot line on the floor of the room is the hypotenuse of a right triangle of leg lengths 12 ft and 20 ft. So, its length is \( \sqrt{12^2 + 20^2} = 23.3 \). The length whose measure we want is the hypotenuse of a triangle \( \triangle ACB \) whose leg lengths are 23.3 and 8 feet. Using the Pythagorean theorem again we conclude that the measure of the line segment in which we
are interested \((AB)\) is \(\sqrt{8^2 + 23.3^2} = 24.64\); since our original data were given in feet, the answer: 25 ft. should suffice.

**Example 8.** What is the length of the longest line segment in the unit cube?

**Solution.** We can use the same figure as in the preceding problem, taking that to be the unit cube. Then the length of the diagonal on the bottom face is \(\sqrt{1^2 + 1^2} = \sqrt{2}\) units, and the length of the diagonal \(AB\) is \(\sqrt{1^2 + (\sqrt{2})^2} = \sqrt{3}\).

**Example 9.** An 18 ft ladder is leaning against a wall, with the base of the ladder 8 feet away from the base of the wall. Approximately how high up the wall is the top of the ladder?

In the accompanying figure, we want to find the value of \(h\) The configuration is a right triangle with hypotenuse (the ladder) of length 18 feet, the base of length 8 feet, and the other leg of length \(h\). By the Pythagorean theorem, we have

\[
h^2 + 8^2 = 18^2 \quad \text{or} \quad h^2 + 64 = 324.
\]

Then \(h^2 = 260\). Since we just want an approximate answer, we look for the integer whose square is close to 260: that would be 16 \((16^2 = 256)\). So, the top of the ladder hits the wall about 16 feet above the ground.

**Example 10.** In the movie *Despicable Me*, an inflatable model of The Great Pyramid of Giza in Egypt was created by Vector to trick people into thinking that the actual pyramid had not been stolen. When inflated, the false Great Pyramid had a square base of side length 100 m. and the height of one of the side triangles was 230 m. What is the volume of gas that was used to fully inflate the fake Pyramid?
Solution. The situation is depicted in figure 13. Now, we know that the formula for the volume of a pyramid is $\frac{1}{3}Bh$, where $B$ is the area of the base and $h$ is the height of the pyramid (the distance from the base to the apex, denoted by $h$ in the figure). Since the base is a square of side length 100 m., its area is $10^4$ m$^2$. To calculate the height, we observe (since the apex of the pyramid is directly above the center of the base), that $h$ is a leg of a right triangle whose other leg is 50 m. and whose hypotenuse has length 230 m. By the Pythagorean theorem $h^2 + 50^2 = 230^2$. Calculating, we find $h^2 = 230^2 - 50^2 = 52900 - 2500 = 50400$. Taking square roots, we have $h = 225$ approximately. Then, the volume of the pyramid is

$$\text{Volume} = \frac{1}{3}(10^4)(2.25 \times 10^2) = .75 \times 10^5 = 75,000 \text{ m}^3.$$  

Section 10.4 The Distance Between Two Points.

Apply the Pythagorean theorem to find the distance between two points. 8G8.

For any two points $P$ and $Q$, the distance between $P$ and $Q$ is the length of the line segment $PQ$.

We can measure the distance between two points with a ruler, and if we are looking at a scale drawing, we will have to use the scale conversion. If the two points are on a coordinate plane, we can find the distance between the points using the coordinates by applying the Pythagorean theorem. The following sequence of examples demonstrates this method, starting with straight measurement.

Example 12.

- a). Using a ruler, estimate the distance between each of the three points $P$, $Q$ and $R$ on figure 12.
The measurements I get are $PQ = 39$ mm; $PR = 39$ mm and $QR = 41$ mm. Of course, the actual measures one gets will depend upon the display of the figure.

- b) Now measure the horizontal and vertical line segments as shown in the figure, and confirm the Pythagorean theorem.

The horizontal line is $30$ mm, and the vertical line is $27$ mm. Now $30^2 + 27^2 = 900 + 729 = 1629$, and $39^2 = 1621$. This approximate values are close enough to confirm the distance calculation, and attribute the discrepancy to variation in measurement.

Example 13. In the accompanying map of the northeast United States, one inch represents $100$ mies. Using a ruler and the scale on the map, calculate the distance between

a) Pittsburgh and Providence

b) Providence and Concord

c) Pittsburgh and Concord

d) Now test whether or not the directions Providence→Pittsburgh and Providence→Concord are at right angles.

Solution. After printing out the map, we measured the distances with a ruler, and found

a) Pittsburgh to Providence: Four and an eighth inches, or $412.5$ miles;

b) Providence to Concord: $15/16$ of an inch, or $93.75$ miles;

c) Pittsburgh to Providence: Four and a quarter inches, or $425$ miles.
d) Since we are only approximating these distances, we round to integer values and then check the Pythagorean formula to see how close we come to have both sides equal. Let us denote these distances by the corresponding letters \(a\), \(b\), \(c\), so that \(a = 413\), \(b = 94\), \(c = 425\). We now calculate the components of the Pythagorean formula:

\[
a^2 = 170,569, \quad b^2 = 8,836, \quad \text{so} \quad a^2 + b^2 = 179,405; \quad c^2 = 180,625.
\]

The error, 1220, is well within one percent of \(c^2\), so this angle can be taken to be a right angle.

**Example 14.** On a coordinate plane, locate the points \(P(3, 2)\) and \(Q(7, 5)\) and estimate the distance between \(P\) and \(Q\). Now draw the horizontal line starting at \(P\) and the vertical line starting at \(Q\) and let \(R\) be the point of intersection. Now calculate the length of \(PQ\) using the Pythagorean theorem.

First of all, we know the coordinates for \(R\): \((7,2)\). So the length of \(PR\) is 4, and the length of \(QR\) is 3. By the Pythagorean theorem, the length of \(PQ\) is \(\sqrt{4^2 + 3^2} = 5\). The measurement with ruler should confirm that.

**Example 15.** Find the distance between each pair of these three points on the coordinate plane: \(P(-3, 2)\), \(Q(7, 7)\) and \(R(2, -4)\)
Solution. In figure above have drawn the points and represented the line joining them by dotted lines. To calculate the lengths of these line segments, we consider the right triangle with horizontal and vertical legs and $PQ$ as hypotenuse. The length of the horizontal leg is $7-(-3) = 10$, and that of the vertical leg is $7-2 = 5$. So

$$|PQ| = \sqrt{10^2 + 5^2} = \sqrt{5^2(2^2 + 1)} = 5\sqrt{5}.$$  

For the other two lengths, use the slope triangles as shown and perform the same calculation:

$$|QR| = \sqrt{11^2 + 5^2} = \sqrt{121 + 25} = \sqrt{146},$$
$$|PR| = \sqrt{5^2 + 6^2} = \sqrt{61}.$$  

Recalling examples 7 and 8 above, we note that we can use the Pythagorean theorem to find the distance between two points in space. Given the points $A$ and $B$, draw the rectangular prism with sides parallel to the coordinate planes that has $A$ and $B$ as diametrically opposite vertices (refer to the diagram for ex 7). Then, as in example 7, the distance between $A$ and $B$ is the square root of the sum of the lengths of the side.

**Example 16.** What is the length of the longest line segment in a box of width 10”, length 16” and height 8”? 

Length $= \sqrt{10^2 + 16^2 + 8^2} = \sqrt{100 + 256 + 64} = \sqrt{428} = \sqrt{4 \cdot 107} = 2\sqrt{107}$
inches, or a little more that 20 inches.

These examples show us that the distance between two points in the plane can be calculated using the Pythagorean theorem, since the slope triangle with hypotenuse the line segment joining the two points is a right angle. This can be stated as a formula, using symbols for the coordinates of the two points, but it is best if students understand the protocol and the reasoning behind it, and by no means should memorize the formula. Nevertheless, for completeness, here it is.

**The Coordinate Distance Formula.** Given points $P : (x_0, y_0)$, $Q(x_1, y_1)$

$$|PQ| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}.$$

Example 17. Figure 15 is a photograph, by Jan-Pieter Nap, of the Mount Bromo volcano on the island Java of Indonesia taken on July 11 2004.

http://commons.wikimedia.org/wiki/File:Mahameru-volcano.jpeg

From the bottom of the volcano (in our line of vision) to the top is 6000 feet. Given that information, by measuring with a rule, find out how long the visible part of the left slope is, and how high the plume of smoke is.
Using a ruler, we find that the height of the image of the volcano is 12 mm, the length of the left slope is 21 mm, and the plume is 10 mm high. Now we are given the information that the visible part of the volcano is 5000 feet, and that is represented in the image by 12 mm. Thus the scale of this photo (at the volcano) is 12 mm : 6000 feet, or 1 mm : 500 feet. Then the slope is $21 \times 500 = 10,500$ feet, and the plume is $10 \times 500 = 5000$ feet high.

**Example 18.** In my backyard I plan to build a rectangular shed that is 12’ by 20’, with a peaked roof, as shown in figure 16. The peak of the roof is 3’ above the ceiling of the shed. How long do I have to cut the roof beams?

![Figure 16](image)

**Example 19.** Figure 17 is a detailed map of part of the Highline Trail, courtesy of www.christine@lustik.com. Using the scale, find the distance from Kings Peak to Deadhorse Pass as the crow flies. Now find the length of the trail between these points. In both cases, just measure distances along horizontal and vertical lens and use P theorem. Measuring with a ruler on the scale at the bottom, we find that the scale is 20 mm : 5 km, or 4 mm/km. Measuring the distance from King’s Peak to Deadhorse Pass, we get 108 mm. Then the actual distance in km is

$$(108 \text{ mm}) \cdot \frac{1 \text{ km}}{4 \text{ mm}} = 27 \text{ km}.$$  

Now, to find the length of the trail, you will have to measure each straight length individually and add the measurements. Alternatively, you can go to http://lustik.com/highline_trail.htm and read an entertaining account of the hike.