Chapter 6. Real World Equations and Inequalities

In this chapter we will use the geometric relationships that we explored in the last chapter and combine them with the algebra we have learned in prior chapters. First we recall the methods developed in (the first section of) Chapter 3, but without the bar models, focusing on algebra as an extension of the natural arithmetic operations we have been performing until now in solving problems. We will then focus on how to solve a variety of geometric applications and word problems. We will also encounter inequalities and extend our algebraic skills for solving inequalities. By this chapter’s end, we will be able to set up, solve, and interpret the solutions for a wide variety of equations and inequalities that involve rational number coefficients.

Introduction

*Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.* 7.EE.4

Consider the sentences $9 + 4 = 13$ and $8 + 6 = 12$. The first sentence is true, but the second sentence is false. Here is another sentence: $\diamond + 9 = 2$. This sentence is neither true nor false because we don’t know what number the symbol $\diamond$ represents. If $\diamond$ represents $-6$ the sentence is false, and if $\diamond$ represents $-7$ the sentence is true. This is called an *open sentence*.

A symbol such as $\diamond$ is referred to as an *unknown* or a *variable*, depending upon the context. If the context is a specific situation in which we seek the numeric value of a quantity defined by a set of conditions, then we will say it is an “unknown.” But if we are discussing quantifiable concepts (like length, temperature, speed), over a whole range of possible specific situations, we will use the word “variable.”

So, for example, if I am told that Maria, who is now 37, three years ago was twice as old as Jubana was then, then I would write down the equation $37 - 3 = 2(J - 3)$, where $J$ is the unknown age of Jubana. But if I write that $C = 2\pi r$ for a circle, $C$ and $r$ are the measures of circumference and radius (in the same units) for *any* circle. In this context, $C$ and $r$ are “variables.”

Various letters or symbols can be used, such as $\diamond, x, y, a, b, c$. Variables are symbols used to represent any number coming from a particular set (such as the set of integers or the set of real numbers). Often variables are used to stand for quantities that vary, like a person’s age, the price of a bicycle, or the length of a side of a triangle. But, in a specific instance, if we write $\diamond + 9 = 2$, $\diamond$ is an unknown, and the number $-7$ makes $\diamond + 9 = 2$ true, so is a correct value of the unknown, and is called a *solution* of the open sentence.

In Chapter 3, we moved seamlessly from pictorial models of arithmetic situations to algebraic formulations of those models, called expressions, without having said what an expression is. Let’s do that now: *an expression* is a phrase consisting of symbols (representing unknowns or variables) and numbers, connected meaningfully by arithmetic operations. So $2x - 3(5 - x)$ is an expression as is $(4/x)(x^2 + 3x)$. 
It is often the case that different expressions have the same meaning: for example \(x+x\) and \(2x\) have the same meaning, as do \(x-x\) and 0. By the same meaning we mean that a substitution of any number for the unknown \(x\) in each expression produces the same numerical result. We shall call two expressions equivalent if they have the same meaning in this sense: any substitution of a number for the unknown gives the same result for both expressions.

Since checking two expressions for every substitution of a number will take a long time, we need some rules for equivalence. These are the laws of arithmetic: \(2x + 6\) and \(2(3 + x)\) are equivalent because of the laws of distribution and commutativity. In the same way, \(2x + 5x\) is equivalent to \(7x\); \(-2(8x - 1)\) is equivalent to \(2 - 16x\), and so forth. Reliance on the laws of arithmetic is essential: to show that two expressions have the same meaning, we don’t/can’t check every number; it suffices to show that we can move from one expression to the other using those laws. Also, to show that, for example \(3 + 2x\) and \(5x\) are not equivalent, we only have to show that there is a value of \(x\) that, when substituted, does not give the same result. So, if we substitute 1 for \(x\) we get 5 and 5. But what if we substitute 2 for \(x\): we get 7 and 10. The expressions are not equivalent.

- To show that two expressions are equivalent, we must show how to get from one to the other by the laws of arithmetic. However, to show two expressions are not equivalent, we need only find a substitution for the unknown that gives different results for the two expressions.

Open sentences that use the symbol ‘\(=\)’ are called equations. An equation is a statement that two expressions on either side of the ‘\(=\)’ symbol are equal. The mathematician Robert Recorde invented the symbol to stand for ‘is equal to’ in the 16th century because he felt that no two things were more alike than two line segments of equal length.

Statements involving the symbols ‘\(>\)’, ‘\(<\)’, ‘\(\geq\)’, ‘\(\leq\)’ or ‘\(\neq\)’ are called inequalities. Let’s review the meanings of these symbols:

- \(x > 3\) means “\(x\) is greater than 3.”
- \(7 < 4\) mens “7 is less than 4.” (Note that these are simply statements, with truth or falsity to be determined.)
- \(x \geq y\) means “\(x\) is greater than or equal to \(y\).”
- \(x \leq x^2\) means “\(x\) is less than or equal to \(x^2\).”
- \(a \neq b + c\) means “\(a\) is not equal to \(b + c\).”

To solve an equation or inequality means to determine whether or not it is true, or for what values of the unknown it is true. Once found, those values are called the solutions of the problem.
Example 1. Find the solutions of these open sentences by inspection. How many solutions are there?

- $\bigvee - 4 = 1$. ($\bigvee = 5$, one solution)
- $x + \frac{1}{4} = \frac{7}{4}$. ($x = \frac{6}{4} = \frac{3}{2} = 1\frac{1}{2}$, one solution)
- $5x = 12$. ($x = \frac{12}{5} = 2\frac{2}{5}$, one solution)
- $2x = 0$. ($x = 0$, one solution)
- $0x = 5$. (Any value of $x$ would still make the left side equal zero and $0 \neq 5$, no solutions exist)
- $x$ is a whole number and $x < 4$. (The solutions are 1, 2, and 3, three solutions)
- $x = 2x - x$. (Any value of $x$ makes this sentence true, there are infinitely many solutions)

Writing mathematics

Variables, equations, and inequalities empower us to write down ideas in concise ways.

Example. Sofia is in third grade and loves mathematics. She was thinking about numbers one day, and she wrote

$$2 + 2 = 4 + 4 = 8 + 8 = 16 + 16 = 32 + 32$$

on her paper and was eager to continue. Apollo was impressed, but confused. “Wait, something’s wrong,” he told her. “$2 + 2$ is not equal to $4 + 4$. ” Sofia agreed and corrected her writing like so

$$2 + 2 = 4. \quad 4 + 4 = 8. \quad 8 + 8 = 16. \quad 16 + 16 = 32. \quad \text{etc...}$$

An equation is a sentence. The symbol ‘$=$’ means ‘is equal to,’ and if you run all your thoughts together, you may say something false.

Writing mathematics has a grammar of its own. Just like a sentence has punctuation marks, mathematics makes use of grouping symbols to help the reader understand the meaning. Equality and inequality symbols are grouping symbols. They divide an equation or inequality into a left hand side (LHS) and a right hand side (RHS). Parentheses and fraction bars are also used as grouping symbols.

To better understand what is meant by expression, equation, and inequality examine the examples in the table below. Note that equations and inequalities are statements, while expressions are more like mathematical phrases.
Equivalent equations

As with expressions, equations are said to be equivalent if they have the same meaning. For expressions, this meant that substitution of a number for the unknown in each expression gives the same answer. For equations, it is the same, but as an equation represents a statement, the criterion is true or false: so, the test is this: if the substitution of a number for the unknown in the two equations always gives the same answer to the question, “True or False,” then the equations are equivalent. Another way of putting this is this: a solution of an equation is a number that, when substituted for the unknown, gives the response “true.” So, to make the definition more precise, two equations are equivalent if they have the same set of solutions.

As with expressions, this is an impossible criterion to apply: we cannot test every number. So, we look for laws of arithmetic that do not change the solution set of an equation. For example, $2x = 10$ and $x = 5$ are equivalent equations: because one equation is double the other, so they have the same solution set. Also, $3(x + 9) = 72$ and $4x - 10 = 50$ are equivalent, because they have the same solution ($x = 15$).

Example 2. Write down the sequence of laws of arithmetic the take us from one equation to the other.

The following equations are equivalent

\[
\begin{align*}
3x + 6 &= 15 \\
3x + 8 &= 17 \\
3x &= 9 \\
12x &= 36 \\
x &= 3
\end{align*}
\]

The above five equations are all equivalent because they all have the same solution ($x = 3$). But we can also see that the equations are equivalent because they are related by operations that do not change the solution set:

- If we add 2 to both sides of equation (1), we obtain equation (2).
- If we subtract 8 from both sides of equation (2), we obtain equation (3).
- If we multiply both sides of equation (3) by 4, we obtain equation (4).
• If we divide both sides of equation (4) by 12, we obtain equation (5).

This example illustrates the most important operations on equations that do not change the solution set. They are:

1. If you add (or subtract) the same number on both sides of an equation, the new equation is equivalent to the original equation.

2. If you multiply (or divide) the same nonzero number on both sides of an equation, the new equation is equivalent to the original equation.

Note that when we ‘multiply or divide both sides of an equation by a number,’ we must apply that operation to every term on both sides of the equation. For example: solve $3x + 9 = 87$.

Our goal is an equation of the form $x = \ldots$, so first let us divide both sides by 3 to get:

$x + 3 = 29$

every term in the equation has been divided by 3. Subtract 3 from both sides to get $x = 26$.

We also could have first subtracted 9 from both sides to get $3x = 78$, and then divided by 3. The order of operations does not matter; what is important is that we employ only operations that can be reversed: if we divide all terms in an equation by 3, we can go back to the original equation by multiplying all terms by 3. If we subtract 9 from both sides of the equation, we can go back by multiplying by 9.

Example 3. Explain why it is important to say nonzero number in the second rule above, but not in the first rule above.

Solution. Suppose you add or subtract 0 from both sides of an equation. This doesn’t change the equation at all. Adding 0 is the reverse of subtracting 0, so these operations can be undone.

Suppose now that you multiply both sides of an equation by 0. For example, if we start with $2x - 7 = 15$, multiplying by 0 gives us $0 = 0$. Certainly a simplification, but not a valuable one: there is no way to go from $0 = 0$ back to $2x - 7 = 15$.

Dividing by zero is not allowed because it simply doesn’t make sense. Recall that division can be thought of as the inverse of multiplication. But we just saw that if we multiply all terms of any equation by 0, we get to $0 = 0$. There is no way of reversing this: to get from the equation $0 = 0$ to any equation.

Finally, if an expression in an equation is replaced by an equivalent expression, then the equations are equivalent. As an example:

$$3(x + 2) = 15$$  \hspace*{1cm} (6)

$$3x + 6 = 15$$  \hspace*{1cm} (7)
are equivalent equations. We refer to this as *simplification* or *substitution*.

**Many paths to the solution**

To ‘solve’ or ‘find solutions’ of a given equation, perform the previously mentioned operations to obtain equivalent equations until the *unknown* is alone on one side of the equation. We also call this process ‘isolating’ the unknown on one side of the equation. It can be helpful to simplify expressions on each side of an equation before or while solving the equation. Let’s look at some examples.

**Example 4:** Solve $4x - 7 = 9$.

**Solution:**

\[
\begin{align*}
4x - 7 & = 9 \\
4x - 7 + 7 & = 9 + 7 \quad \text{(Add 7 to both sides)} \\
4x & = 16 \\
\frac{1}{4}(4x) & = \frac{1}{4}(16) \quad \text{(Multiply both sides by $\frac{1}{4}$)} \\
x & = 4
\end{align*}
\]

Therefore, the solution is 4.

Every time we write a new line, we have an algebraic reason for doing so. Recall that multiplying by $\frac{1}{4}$ is the same as dividing by 4.

Recall that in chapter 3 we learned various properties of addition and multiplication. It is wise to study the examples here and see how those properties appear in each step as we solve equations.

**Example 5.** Solve $\frac{3}{5} (x - \frac{5}{6}) = \frac{7}{4}$.

**Solution:**

\[
\begin{align*}
\frac{3}{5} \left(x - \frac{5}{6}\right) & = \frac{3}{4} \\
5 \cdot \frac{3}{5} \left(x - \frac{5}{6}\right) & = 5 \cdot \frac{3}{4} \quad \text{(Multiply both sides by $\frac{5}{3}$)} \\
(x - \frac{5}{6}) & = \frac{5}{4} \\
x - \frac{5}{6} + \frac{5}{6} & = \frac{5}{4} + \frac{5}{6} \quad \text{(Add $\frac{5}{6}$ to both sides)} \\
x & = \frac{15}{12} + \frac{10}{12} = \frac{25}{12}
\end{align*}
\]

Therefore, the solution is $\frac{25}{12}$ or $2\frac{1}{12}$.
Another solution method:

\[
\frac{3}{5} \left( x - \frac{5}{6} \right) = \frac{3}{4}
\]

\[
\frac{3}{5}x - \frac{3 \cdot 5}{5 \cdot 6} = \frac{3}{4} \quad \text{(Apply the distributive property on the LHS.)}
\]

\[
\frac{3}{5}x - \frac{1}{2} = \frac{3}{4} \quad \text{(Simplify the expression } \frac{3 \cdot 5}{5 \cdot 6} \text{)}
\]

\[
\frac{3}{5}x - \frac{1}{2} + \frac{1}{2} = \frac{3}{4} + \frac{1}{2} \quad \text{(Add } \frac{1}{2} \text{ to both sides.)}
\]

\[
\frac{3}{5}x = \frac{3}{4} + \frac{2}{4} \quad \text{(Add } \frac{1}{2} \text{ to both sides.)}
\]

\[
\frac{3}{5}x = \frac{5}{4}
\]

\[
\frac{5}{3} \cdot \frac{3}{5}x = \frac{5}{3} \cdot \frac{5}{4} \quad \text{(Multiply } \frac{5}{3} \text{ to both sides.)}
\]

\[
x = \frac{25}{12}
\]

Again the solution is \(\frac{25}{12}\) or \(2\frac{1}{12}\).

**Example 6.** This process works even with tricky examples. Here we have positive and negative decimal coefficients \(0.2x - 0.4 = -3.4\).

**Solution:**

\[
0.2x - 0.4 = -3.4
\]

\[
0.2x - 0.4 + 0.4 = -3.4 + 0.4 \quad \text{(Add 0.4 to both sides.)}
\]

\[
0.2x = -3.0
\]

\[
\frac{1}{0.2}(0.2x) = \frac{1}{0.2}(-3.0) \quad \text{(Multiply both sides by } \frac{1}{0.2}.\text{)}
\]

\[
x = \frac{-3.0}{0.2} = -\frac{30}{2} = -15
\]

Therefore, the value of \(x\) is \(-15\).

In the next to last step, we could have said we divided both sides by 0.2. This is the same as multiplying both sides by \(\frac{1}{0.2}\), the multiplicative inverse of 0.2. Notice also that \(\frac{1}{0.2} = \frac{10}{2} = 5\). So, another way to finish solving this equation is by multiplying both sides by 5.

Another solution method for this same example:

\[
0.2x - 0.4 = -3.4
\]

\[
2x - 4 = -34 \quad \text{(Multiply both sides by 10, now the decimals are removed.)}
\]

\[
2x = -30 \quad \text{(Add 4 to both sides.)}
\]

\[
\frac{2x}{2} = \frac{-30}{2} \quad \text{(Divide both sides by 2.)}
\]

\[
x = -15
\]
There are many ways to get to the right solution. It could be advantageous to multiply each side of an equation by a number just to remove fractions and decimals.

An equation of the form \( px + q = r \), where \( p, q, \) and \( r \) are any numbers, is called a first order equation. All of the examples illustrated here are equations that can be written in this form (with \( p \neq 0 \)).

**Evaluating formulas**

A formula tells us information about how different variables relate to each other (the plural form of ‘formula’ is ‘formulae’). For example, the perimeter of a polygon is the sum of the lengths of its sides. Perimeter is usually denoted by the letter \( P \). For a triangle of side lengths \( a, b, c \), we express this by the formula \( P = a + b + c \). So, if we are given a triangle of side lengths 19, 7, 21 units, we calculate perimeter as follows:

\[
P = a + b + c, \quad a = 19, \quad b = 7, \quad c = 21
\]

\[
P = 19 + 7 + 21 = 47
\]

Look over the following table of mathematical formulae for the areas and perimeters of geometric objects. Remember that if length is given in certain units (ft., cm., ...), then perimeter is measured in the same units, while area is measured in square units (sq. ft., sq. cm., ...).

For example, if we have a rectangular lot that measures 72 yards in length and 72 yards in width, then to fence in the whole lot we need to calculate the perimeter and use the formula \( P = 2l + 2w \), with \( l = 72 \) yards and \( w = 40 \) yards: \( P = 2(72) + 2(40) = 224 \) yards. To find the area of this lot we use the formula \( A = l \cdot w \), so it has area \( A = 72 \cdot 40 = 2880 \) sq. yds.
Section 6.1. Write and Solve Equations to Find Unknowns in Geometric Situations

Use facts about supplementary, complementary, vertical, and adjacent angles to write and solve multi-step problems for an unknown angle in a figure. 7.G.5

In the last chapter, we learned many relationships among angles that can be expressed with equations.

- The sum of the measures of interior angles in a triangle is $180^\circ$.
- Vertical angles have equal measure.
- Complementary angles add up to $90^\circ$. 

<table>
<thead>
<tr>
<th>Diagram</th>
<th>Formula</th>
<th>Area Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Rectangle" /></td>
<td>$P = 2l + 2w$</td>
<td>$A = l \cdot w$</td>
</tr>
<tr>
<td><img src="image" alt="Triangle" /></td>
<td>$P = a + b + c$</td>
<td>$A = \frac{1}{2}b \cdot h$</td>
</tr>
<tr>
<td><img src="image" alt="Parallelogram" /></td>
<td>$P = b_1 + b_2 + s_1 + s_2$</td>
<td>$A = \frac{1}{2}(b_1 + b_2) h$</td>
</tr>
<tr>
<td><img src="image" alt="Circle" /></td>
<td>$C = 2\pi r$</td>
<td>$A = \pi r^2$</td>
</tr>
</tbody>
</table>
• Supplementary angles add up to 180°.

Using algebraic language can help us to write those relationships and can also help us solve problems. Suppose we knew a few things about the angles in this library, and wanted to know other angles.

![Figure 1]

As in the last examples of chapter 5, we could set up and solve equations for some of the angles, knowing others.

**Example 7.** In the library shown in figure 1, the pitch of the roof is 22° and the angle between the joist and the roof support beam is 38°. Find the measure of the remaining angles of the wooden support structure.

**Solution.** We have used some technical language in phrasing this problem, so let’s spend a moment on that. First, let us make a scale drawing of the roof support structure.

The *pitch* of the roof is the angle it makes with the horizontal. So we have put a horizontal hashed line at the peak of the roof and indicate the angle with measure 22°. A *roof support beam* is a beam from a wall of the structure that makes a diagonal with the roof; in our case, this is the beam represented by the line segment \(AB\). A *joist* is a horizontal floor beam (even if there is no floor); designated in our diagram by the segment \(AC\), and so we have put the angle measure 38 in that place.b Finally, although it has not been made explicit, we
assume that the walls are perfectly vertical: that is, the line segment $AD$ is vertical. With that clarification of the builder’s jargon, we can now calculate the other angles. Since $AD$ is vertical and the hashed line is horizontal, the angle $\angle ADB$ is complementary to the $22^\circ$ angle, and so $\angle ADB = 90 - 22 = 68^\circ$. Also, $\angle DAB$ is complementary to $\angle BAC$ whose measure is $38^\circ$, so $\angle DAB$ has measure $52^\circ$. Using the fact that the sum of the angles of a triangle is $180^\circ$, we find the measure of $\angle DBA$:

$$\angle DBA = 180 - (68 + 52) = 60^\circ.$$
Example 8. In \( \triangle ABC \) of figure 3, we know that \( m(\angle A) + m(\angle B) + m(\angle C) = 180^\circ \), as this is true of all triangles. We also know that \( \angle C \) measures twice the amount of \( \angle B \) since \( m(\angle B) = x \) and \( m(\angle C) = 2x \). We can use the information given to set up and solve an equation. We will find the measure of \( \angle B \) by finding the value of the unknown \( x \).

\[
\begin{align*}
x + 2x + 66 &= 180 \\
3x + 66 &= 180 \quad \text{(Combine like terms.)} \\
3x + 66 - 66 &= 180 - 66 \quad \text{(Add the additive inverse of 66 to both sides.)} \\
3x &= 114 \\
3x \cdot \frac{1}{3} &= 114 \cdot \frac{1}{3} \quad \text{(Multiply the multiplicative inverse of 3 to both sides.)} \\
x &= \frac{114}{3} = 38
\end{align*}
\]

Therefore, \( x = 38 \), so \( m(\angle B) = 38^\circ \). We also can find that \( m(\angle C) = 2 \cdot 38^\circ = 76^\circ \).

Example 9. Here is another diagram suggesting an equation.

It is clear from this picture (figure 4) that \( \angle CAB \) and \( \angle BAD \) are supplementary. Because of this we can write an equation, \( m(\angle CAB) + m(\angle BAD) = 180^\circ \). And again, the diagram
shows algebraic expressions for these angles so we can find the unknown $x$.

\[
2x - 9 + \frac{1}{4}x = 180
\]

\[
\frac{8}{4}x + \frac{1}{4}x - 9 = 180
\]

\[
\frac{9}{4}x - 9 = 180 \quad \text{(Combine like terms.)}
\]

\[
\frac{9}{4}x - 9 + 9 = 180 + 9 \quad \text{(Add the additive inverse of the constant term to both sides.)}
\]

\[
\frac{9}{4}x = 189
\]

\[
\frac{9}{4}x \cdot \frac{4}{9} = 189 \cdot \frac{4}{9} \quad \text{(Multiply the multiplicative inverse of \( \frac{9}{4} \) to both sides.)}
\]

\[
x = \frac{4 \cdot 189}{9} = 84
\]

Therefore, $x = 84$.

Now we can also find that $m(\angle CAB) = (2 \cdot 84 - 9)^\circ = (168 - 9)^\circ = 149^\circ$ and $m(\angle BAD) = (84/4)^\circ = 21^\circ$.

Furthermore, note that since $\angle ADB$ is a right angle, we can also find the measure of $\angle ABD$, since $\angle ABD$ and $\angle BAD$ are complementary:

\[
m(\angle ABD) + m(\angle BAD) = 90.
\]

Since we already found that $m(\angle BAD) = 21^\circ$, we can subtract $\angle BAD$ from the LHS and 21 from the RHS to get $m(\angle ABD) = 69$.

**Example 10.** The perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?

Since we know that the perimeter of a rectangle is equal to 2 times the length plus 2 times the width, we can start by writing the formula $P = 2l + 2w$. In this example the perimeter is 54 cm and the length is 6 cm, so we can put that information in our equation and we will have a new equation with just one unknown quantity, $w$:

\[
54 = 2(6) + 2w
\]

Now, we can proceed with solving the equation as follows:

\[
54 = 2(6) + 2w
\]

\[
54 = 12 + 2w
\]

\[
54 - 12 = 12 + 2w - 12
\]

\[
42 = 2w
\]

\[
42 = 2w
\]

\[
\frac{2}{2} = \frac{2}{2}
\]

\[
21 = w
\]
We have found that \( w = 21 \), so the width of the rectangle must be 21 cm.

**Example 11.** A trapezoid has a perimeter of 47 cm. and an area of 132 sq. cm. The longer of the two parallel sides has length 13 cm and the height of the trapezoid is 11 cm. a) What is the length of the shorter of the two parallel sides?

**Solution.** We refer to the table of formulae above: the perimeter of the trapezoid is given by the formula

\[
P = b_1 + b_2 + s_1 + s_2
\]

where the \( b \)'s refer to the parallel sides, and the \( s \)'s refer to the other sides. Putting information into the formula gives us the equation

\[
47 = 13 + b_1 + s_1 + s_2,
\]

and we want to find \( b_1 \). But we don't know \( s_1 \) and \( s_2 \); what we do know is the height of the parallelogram, but that does not tell us the length of the sides (draw several parallelograms with the given dimensions but with differing lengths for \( s_1 \) and \( s_2 \). Maybe the area formula gives us some hope:

\[
A = \frac{1}{2}(b_1 + b_2)h.
\]

Putting in the given data gives us the equation: \( A = 132 = \frac{1}{2}(13 + b_2)(11) \), which we can solve for \( b_2 \): Multiply both sides by 2 and divide both sides by 11 to get \( 22 = 13 + b_1 \), to find \( b_2 = 9 \).

**Section 6.2. Write and Solve Equations from Word Problems**

*Solve word problems leading to equations of the form \( px + q = r \) and \( p(x + q) = r \), where \( p \), \( q \), and \( r \) are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach.* 7EE4a.

**A note about units**

In chapter 3 we stressed that, in expressing a proportion, it is necessary to be precise about units: the ratio of volume of water to weight might be 1:8 or 1:2 - in the first case we should read thus as "one cup to 8 ounces" and in the second, "one quart to 2 pounds." And proper use of units was seen to be essential to understanding scaled drawing. And the nature of units is necessary to understanding equations: when we write is a statement that two things are equal, the units on each side of an equation must be the same. To answer the question, "how many cups of water makes 32 ounces?", we write \( 8x = 32 \), where \( x \) is the number of cups but what is being equated is number of ounce.

In fact, concentration on the relevant units in a problem may also help us to write an equation. For example, suppose someone drove from Layton to Salt Lake City in 30 minutes. If they traveled at a constant speed, how fast were they driving? Typically we talk about driving speeds by the units of miles per hour. But in this case, we are just given minutes.
Hence, it may be helpful to write 30 minutes as $\frac{1}{2}$ hour. Since we want to know an answer in miles per hour, we see that we are missing some information here. We need to know how many miles it is from Layton to Salt Lake City. A map shows that it is about 24 miles. So, if we let $s$ be the speed in miles per hour, we can write the equation

$$s = \frac{24}{\frac{1}{2}}.$$ 

The units match on both sides of the equation, so we don’t need to write them down. We can perform arithmetic without them: $s = \frac{24}{\frac{1}{2}} = 48$. And in conclusion, to report our answer we use the units again. The speed was 48 miles per hour.

**Example 12.** The youth group is going on a trip to the state fair. The trip costs $52. Included in that price is $11 for a concert ticket and the cost of 2 passes, one for the rides and one for the game booths. Each of the passes cost the same price. Write an equation representing the cost of the trip and determine the price of one pass.

We start by identifying an unknown and use a variable to represent it. Since we want to determine the price of one pass, we will say $x$ is the price of one pass.

Next, we identify what we do know. We know that the cost of the trip is $52. Both of these steps require careful reading of the problem, and translation of these ideas into mathematical language.

What connects the two pieces of information together? Well, the fact that the $52 trip covers the cost of an $11 item and two passes. That is: $52 is equal to $11 plus the cost of two passes. Well, we have already represented the cost of each pass by $x$, so this brings us to the equation: $52 = 11 + 2x$. It is the algebraic expression of the statement: $52$ is the cost of an $11 item and 2 passes.

This diagram shown is also a helpful way to arrive at the given equation.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$x$</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>52</td>
</tr>
</tbody>
</table>

To solve this equation we may make algebraic steps to isolate $x$ on one side, as follows

$$2x + 11 = 52$$
$$2x = 41$$
$$x = 20.5$$

Now that we have solved the equation, we interpret the result. The price of one pass must be $20.50.$
Does that result make sense? Sure it does, because 2 passes at that price would be $41. Plus $11 for the concert ticket and we’ve got the total price of $52. Notice how this reasoning is represented and expressed in algebraic equations.

The above examples could also be solved by simple arithmetic reasoning. Take the last example: Of $52, $11 was spent on the concert ticket, so the two passes together cost $52-$11 = $41, and each cost half that, or $20.50. In fact, the algebraic method encodes this reasoning.

The student may ask, “why are we learning algebraic methods, when the arithmetic method is so easy?” There are many answers to this question. First, the arithmetic method looked easy because the arithmetic was easy. Consider this:

**Example 13.** Our neighborhood group went on an outing at an amusement park. There were 93 people, of whom 38 were children. The price for an adult admission was $25.50. The group leader paid the whole bill at $1925.00. What is the price of a child’s ticket?

An example like this illustrates the value of having a systematic way of encoding the information and data of a given problem in a way that leads to a straightforward algorithm for solving the problem. If this example does not suffice, I know that the teacher will be able to easily come up with more convincing examples. However, a much more compelling answer to the quest, “why algebra” is that once we understand how to analyze and encode the information in a problem, we can apply those ideas to harder problems, and soon, we are able to solve problems with millions of data points and thousands of unknowns (as in internet search engines). More precisely, we are able to give exact instructions to the computer so that it can solve millions of such problems in seconds, as millions of users press the ”search” button.

The first problem above was one for which we were initially not given enough information, so we were unable to solve it. For such problems, algebra gives us a technique for discovering that some information is missing.

Some general guidelines for solving word problems are:

1. Identify what is unknown and needs to be found. Represent this with a variable.
2. Write an equation that expresses the known information in terms of the unknown.
3. Determine whether or not you have enough information to solve the equation. If so, do so.
4. Interpret the solution and ensure that it makes sense.

**Section 6.3. Solve and Graph Inequalities, Interpret Inequality Solutions**

Solve word problems leading to inequalities of the form $px + q > r$ or $px + q < r$, where $p$, $q$,
and \( r \) are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem. 7EE4b.

In grade 6 students began to work with inequalities between numbers, here we first review the symbols representing inequalities, and then we explore how to work to simplify inequalities between expressions.

Recall, from the discussion on equivalence at the beginning of this chapter: for \( a \) and \( b \) two numbers

- \( a < b \) means “\( a \) is less than \( b \),” as in \( \frac{1}{2} < \frac{3}{4} \).
- \( a \leq b \) means “\( a \) is less than or equal to \( b \),” as in \( \frac{10}{16} \leq \frac{5}{8} \).
- \( a > b \) means “\( a \) is greater than \( b \),” as in \( 11 > 4 \).
- \( a \geq b \) means “\( a \) is greater than or equal to \( b \),” as in \( 11 \geq \frac{22}{2} \).

The realization of numbers as points on the line gives another very useful interpretation of these symbols: on the real line \( a < b \) means that \( a \) lies to the left of \( b \), and \( a \leq b \) means that \( a \) lies to the right of \( b \).

Consider the statement \( x < 4 \). There isn’t just one value of \( x \) that makes this open sentence true. The inequality would be true if \( x = 3 \) or if \( x = 3.9 \) or if \( x = 0 \) or \( x = -2 \) or \( x = \frac{1}{8} \). In fact, it is impossible to list all the values that make this statement true. The solution set for this open sentence contains an infinite number of values. We can resolve this issue by representing solutions of inequalities graphically on the number line.

To represent a single number on a number line, we draw a filled-in dot on that number. Here is a graph of \( x = 4 \).

\[
\begin{array}{c}
-4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

So, suppose we have

\[9x - 10 = 26.\]

We can use algebra to find that \( x = 4 \) is the solution, and then draw a picture like the one above to show that the solution set consists of the single point \( x = 4 \).

To show the solution set of the inequality \( x \leq 4 \) on a number line, we fill in the point \( x = 4 \) and shade the region on the number line less than 4 like so:
The arrow indicates that the shading should continue forever, so numbers like \(-4\) and even \(-100\) are included.

Now, suppose we are given the relation

\[
9x - 10 \leq 26.
\]

If we add 10 to both sides of the inequality, then, in the real line representation, everything is shifted to the right by ten units, so we have the relation \(9x \leq 36\), which has the same solution set. Now, if we divide by 9, we are just changing the scale by a factor of \(1/9\), so again we have the same solution set, but now written as \(x \leq 4\), with the same graph on the real line as above. Thus this is the graph of the solution set of the inequality \(9x - 10 \leq 26\).

Suppose now that we want to show \(x < 4\) on the number line. We can read this inequality as \(x \text{ 'is less than'}\ 4\) or \(x \text{ 'is strictly less than'}\ 4\) to emphasize that \(x = 4\) is not part of what we’re talking about. On the number line we represent that like this:

\[
\begin{array}{cccccccccc}
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
-4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ }
\end{array}
\]

Notice that the dot over the number 4 is left open to show that the number 4 is not included in the set of values that makes \(x < 4\) true.

For the most part we can work with inequalities in much the same way that we work with equations. But there are some important differences too.

As with equations, two inequalities are equivalent if they have the same solution set. When we work with equations, we can add (or subtract) the same number to both sides of the equation, and we can multiply (or divide) the same nonzero number to both sides of the equation, and the result is an equivalent equation. Do these same rules apply to inequalities? Let’s go through this carefully, using the real line representation of the number system.

- Add a number to both sides of an inequality: the solution set does not change. If the number is positive, the effect is to shift everything to the right by that number - by “everything” we mean both sides of the inequality. So if \(a < b\), the \(a + 10 < b + 10\), and \(a - 3 < b - 3\), and so, if \(E\) and \(F\) are expressions, the solution set of \(E < F\) is the same as the solution set \(E + 10 < F + 10\) and \(E - 3 < F - 3/\), and so forth for any number replacing 10 and 3.

- Multiply both sides of an inequality by a positive number \(a\): the solution set does not change. We can think of this as a rescaling: when we replace every number \(x\) by \(ax\) we are just replacing the unit length 1 by the unit length \(a\). So the relation between two numbers remains the same, and so the solution set of \(E < F\) is the same as the solutions set of \(aE < aF\).]
Multiply both sides of an inequality by -1 and reverse the inequality: the solution set does not change. Recall that multiplication by -1 takes any point on the line to the point on the other side of 0 of the same distance from 0. So, 7 is two units further away from 0 as is 5, so the same must be true of -7 and -5. But whereas 7 is 2 units to the right of 5, -7 (being two units further away from 0) must be 2 units to the left of 5. Consider the statement: \(-x > 3\). This decrees the set of all points whose opposite is greater than 3. Now if \(-x\) is to the right of 3, then \(x\) is to the left of -3; that is \(x < -3\). So, we see, using the number line, that multiplication by -1 changes left of to right of and right of to left of. \(5 < 7\) says that 7 is two units to the right of 5, and \(-7 < -5\) says that -7 is two units to the left of five. In general, if \(a < b\), then \(-a > -b\); that is, for any \(a\) and \(b\), both statements are true at the same time, or false at the same time. So, this is true of expressions: \(E < F\) and \(-E > -F\) have the same solutions set: if a number is inserted for the unknown in these expressions, the two statements are either both true or both false.

**Example 14.** Solve and graph the solution set of the following inequality:

\[
8(10 - x) < 20
\]

\[
(10 - x) < \frac{20}{8}
\]

\[
10 - x < \frac{5}{2}
\]

\[
10 - x + (-10) < \frac{5}{2} + (-10)
\]

\[
-x < \frac{5}{2} - \frac{20}{2}
\]

\[
-x < -\frac{15}{2}
\]

\[
x > \frac{15}{2}
\]

And the solution set looks like:

![Solution Set Graph](image)

**Example 15.** As a salesperson, you are paid $50 per week plus $3 per sale. This week you want your pay to be at least $100. Write an inequality for the number of sales you need to make, and describe the solutions.

Let \(x\) be the number of sales made in the week. This week you want,

\[
50 + 3x \geq 100.
\]
Solving this,

\[ 50 + 3x \geq 100 \]
\[ 3x \geq 50 \]
\[ x \geq \frac{50}{3} = 16\frac{2}{3} \]

So, the conclusion is you will need to make at least \(16\frac{2}{3}\) sales this week, and since \(\frac{2}{3}\) of a sale isn’t possible, you will need to make at least 17 sales to exceed $100 of earnings.

**Example 16.** A triangle has a perimeter of 20 units. One side is 9 units long. What is the length of the shortest side?

Let \(a\) be the variable to represent the shortest side. Here is a picture.

We know that \(a > 0\) because it is a length. Also it is true that \(a < 9\) since \(a\) is the shortest side. We will see later, that this is not so important.

If the perimeter of the triangle is 20 units, then we must have \(a + 9 + b = 20\). So the third side \(b\) must be \(20 - 9 - a\) units, or \(11 - a\) units.

Since \(a\) is the shortest side, it must be less than the third side, \(11 - a\). So,

\[
\begin{align*}
a & < 11 - a \\
a + a & < 11 - a + a \\
2a & < 11 \\
a & < \frac{11}{2}
\end{align*}
\]

If \(a\) is less than \(\frac{11}{2} = 5.5\), then it automatically is also less than 9 units. So this condition that \(a < \frac{11}{2}\) is a stronger restriction than \(a < 9\).

In conclusion, \(a > 0\) and \(a < \frac{11}{2}\). We can write this as \(0 < a < \frac{11}{2}\), and it means the value of \(a\) lies between 0 and \(\frac{11}{2}\).
Summary

To solve a given equation, we perform a sequence of steps that will replace the equation with an equivalent equation, such that the unknown is isolated on one side of the equation. The following actions result in equivalent equations and may be performed to solve an equation:

- Add/subtract the same quantity on both sides of the equation.
- Multiply/divide the same nonzero quantity on both sides of the equation.
- Replace an expression or quantity in an equation by an equivalent expression or quantity.

The process of setting up and solving equations is an art. There is not just one way to get to the solution of an equation. Many routes are possible provided each algebraic step is justifiable.

The same process works for solving inequalities. We must remember however, that if we multiply both sides of an inequality by a negative number, the direction of the inequality reverses.

Equations and inequalities arise in a variety of contexts in life including financial problems and geometry. If we carefully read the problem and write a corresponding equation or inequality, the properties of numbers and operations can help us do the rest of the work to solve many kinds of problems.