Chapter 5
Functions

In the preceding chapters we started to move our thinking about equations from *looking for a solution* to that of *expressing a relation* and of the use of letters to represent *unknown numbers* or *quantities* to that of *variables*. In both those cases we thought of x (or y or z or . . .) as a yet-to-be-determined number (or numbers) to be found by "solving" in the case of "unknowns", and "measuring" in the case of "quantities." But now we interpret the symbols x, y, z as *variables*; that is, they are to be understood as ranging over a whole set of numbers, and our interest in those variables is in understanding the relation expressed by the equation. As we shall see, this is not so hard if the relation is expressed as a graph, harder if expressed algebraically or by a table, and difficult if expressed by an algorithm. In all cases, we are moving from a *static* study of relations to a *dynamic one*: it is in this sense that letters represent "variables."

An equation with two variables x, y expresses a relationship between them. A solution of the equation consists of two specific numbers, one for each variable, which, when substituted in the equation makes a true statement. In case there is more than one solution, we may talk about *the solution set*. We usually use an ordered pair (x, y) to represent each solution. The order indicates which variable represent which number. Thus, the instruction "substitute $(5, -1)$ in the equation" means: set $x = 5$ and $y = -1$. For example, if the relation is $3x - 2y = 1$, then $(1, 1)$ is in the relation, but $(2, 3)$ is not.

In section 2 of chapter 3 we defined *relation* and *function* in rather abstract terms, and went on to illustrate by specific examples. In particular a *function* (written $y = f(x)$ and expressed as "y is f of x") is a set of instructions which produce, from a choice of specific number for x (called the input), a specific value for y (the output). Said another way, in any function, a given input does not give one output some of the time and a different output at other times. In this chapter, our focus will be on the relation between inputs and outputs, and not on the set of instructions that produce an output for a given input. To say this a different way, our interest is not on the details of the calculation of a y when given an x, but rather on questions like: if x gets larger, what happens to y? If x is halved, what happens to y? If x is replaced by $x + 2$, what happens to y? This is why we introduce the letter "f" to represent the set of instructions, without focusing on them. So, we can read "$y = f(x)$" as "start with x, do f to it, and record the output y."

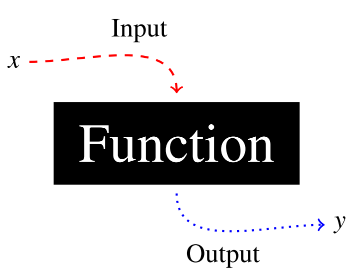

Figure 1

We now look at a function as a "black box" as in figure 1, so that we can concentrate on the relation between input and output, and move away from the mechanics of computing values of a function.

This chapter completes this transition from the concept of *unknown* to that of *variable*, and that from *equation* to that of *function*. We will focus on characteristics that separate linear from nonlinear functions. In the last section we discuss, in contexts, ways of expressing the relation among variables through various representations.

Section 5.1 What is a Function?

1. Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.

2. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). 8.F.1,2

Example 1.

The following table is that of the bus schedule between Salt Lake City and Price.

Salt Lake City to Price										
LvSLC	8:00	9:00	10:30	12:00	13:00	14:30	16:00	17:00	18:30	20:00
ArrPrice	11:15	12:15	13:45	15:15	16:15	17:45	19:15	20:15	21:45	23:15

Examining the table, we notice several things: first of all, it takes the 8:00 bus three hours 15 minutes to make the trip; furthermore, this time is the time every trip takes. Also, the change between any two departure times is the same as the change between any two arrival times. Graphing these data (see Figure 2) makes these observations even more clear. The graph shows that the data lie on a line.

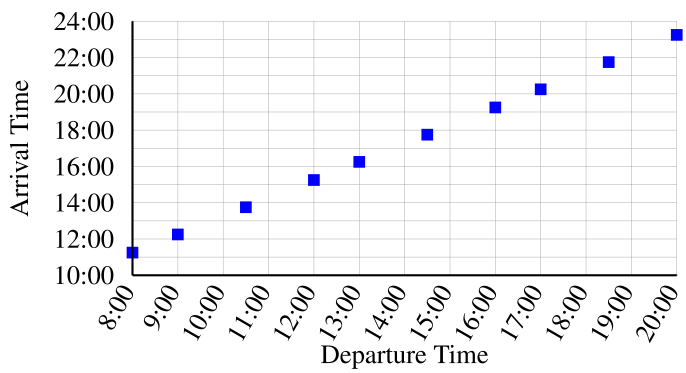

Figure 2. Bus Schedule

In fact, it is a line of slope 1, since any change in departure time results in precisely the same change in arrival time. We conclude that, whenever a bus leaves Salt Lake City, it arrives in Price 3 hours and 15 minutes later. This we call a *model* for the given data: in this case the model is linear. We use the model to show us immediately when a bus arrives in Price, for *any* given departure time from Salt Lake City.

If a new bus, with departure time 6:30 were to be added to the schedule, we schedule it to arrive in Price at 9:45. More generally, if a bus leaves SLC at D o'clock, it should be expected to arrive in Price at $D + 3 : 15$ o'clock. Letting A represent the arrival time, we arrive at this relationship between D and A: $A = D + 3 : 15$. We see that this formula tells us that the arrival time is completely determined by the departure time; that is A is a function of D. In such a statement, we consider A and D as variables in the sense that they can have any (time) value, and the relation $A = D + 3 : 15$ will hold.

Before going on, we note that, in the real world, arrival time is not completely determined by departure time; factors along the road may delay, or advance, the arrival of the bus. Figure 3: Real Data gives a more realistic graph of what may actually happen in a day.

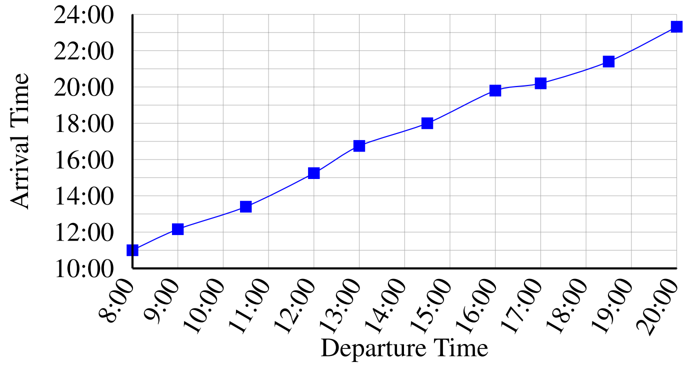

Figure 3

This graph, of actual data, does give us important information: we should expect, on average, for the trip to take 3 hours and 15 minutes. However, in the early morning and late evening, the trip is likely to be quicker, while in the late afternoon, it is likely to take longer. We can still model these data with the straight line $A = D + 3 : 15$, but with the understanding that the arrival time is subject to traffic and weather conditions. We shall return to the subject of models for real data in Chapter 6. Our goal there will be to interpret tables of actual data so as to discover a curve, or a formula, that best models the actual data. For now, let us consider relations between two variables that give rise to functions.

> **Function Concept**: Given two variables, x and y, we will say that y is a *function* of x if there is a set of instructions (which may be expressed as a formula, algorithm or recipe) that determine a specific y for a given x.

The notation used to assert that y is a function of x is $y = f(x)$, where f stands for the set of rules that tell us how to go from x to y. We read $y = f(x)$ as "y equals f of x. Of course, we may use other letters (such as g, h, etc). to represent other functions. This notation sometimes causes confusion, for students have become used now to using letters to represent numbers. So it is useful at this time to be clear that, for the first time, we are using a letter (f) to represent an *action* (implementing a set of rules) rather than a number.

Example 2.

$$y = 3x + 7$$

This can also be given by the set of instructions: pick a number x, multiply it by 3 and add 7. Notice that the instructions clarify the order of operations much better than the formula does, so it is good practice to translate formulas to sets of instructions to better understand them - in fact this is exactly what happens when we execute a sequence of operations on a calculator.

Example 3.

$$y = \frac{1}{x}, \quad x > 0 .$$

Here, we do not have a rule to give a value of y corresponding to $x = 0$. We say that the function is *not defined* for $x = 0$, or 0 is not in the *domain* of the function. For this function, x is a positive number, so we often make explicit that we are only interested in the function for positive values of x. For this function, we say that x and y are *inversely proportional* in the sense that if x is multiplied by any number, the y is divided by that number.

Plotting a set of values (x, y) that are related by a function provides a useful visualization of the function.The usefulness depends upon the extent to which the selected points illustrate the important features of the function. So, given rules describing a function, we create a set of points (x, y), where x is a number to which the rules apply (that is, x is in the domain of the function) and y is the number we get when applying the rule . When we plot enough points, we join them with a curve to get a representation of the function. For the general function this may take some skill or additional information contained in the context, but - as we have seen in the preceding chapter - for a linear function we need only find two points on the graph, and connect them with a line.

Let's go through this analysis for the above examples.

Example 2 revisited.

$y = 3x + 7$

Make a table of representative values

x	-2	-1	0	1	2	3	4
y	1	4	7	10	13	16	19

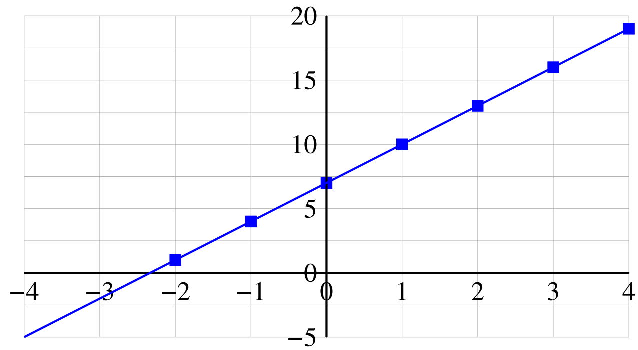

Figure 4

Example 3 revisited.

$$y = \frac{1}{x}, \quad x > 0 .$$

The rule here is "pick a positive number and take its multiplicative inverse." We create the table using the first eight positive half integers:

x	.5	1	1.5	2	2.5	3	3.5	4
y	2	1	.67	.5	.4	.33	.29	.25

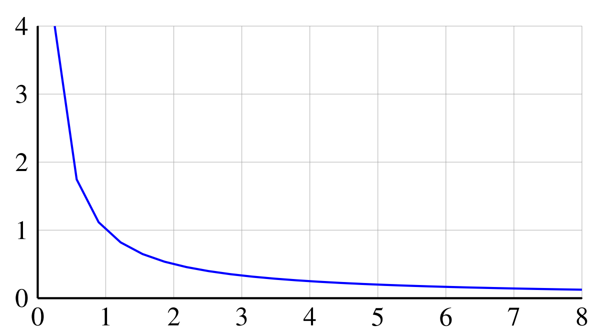

Figure 5

EXAMPLE 4.

The "Rambo Fliers" Come to Town. The local arena in Smalltown, Wa., with occupancy limit 6000, hosted the fabulously popular drums and celestine group, the "Rambo Fliers" for a performance at 7 pm one Saturday night. Admission was at a fixed price, and doors opened at 4 pm with open seating. To accommodate those searching, the audience was admitted in groups (all those waiting to get in) every 15 minutes. The graph below is of the audience count at 15 minute intervals from 4 pm to 7 pm.

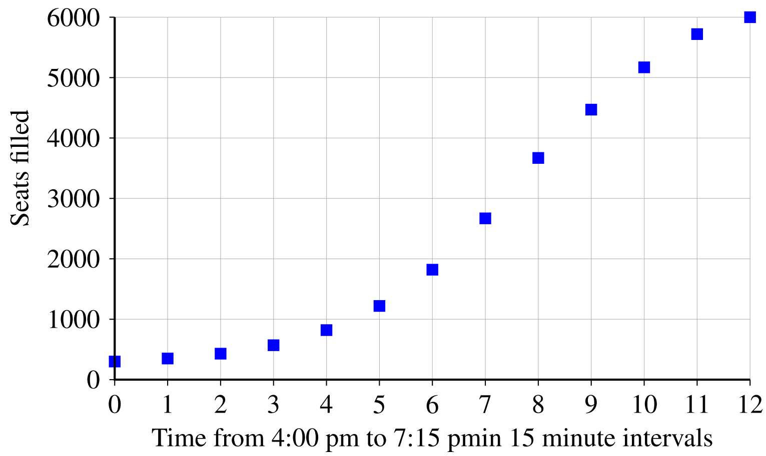

Figure 6. Occupancy at an Event

If we connect these lines, we get this:

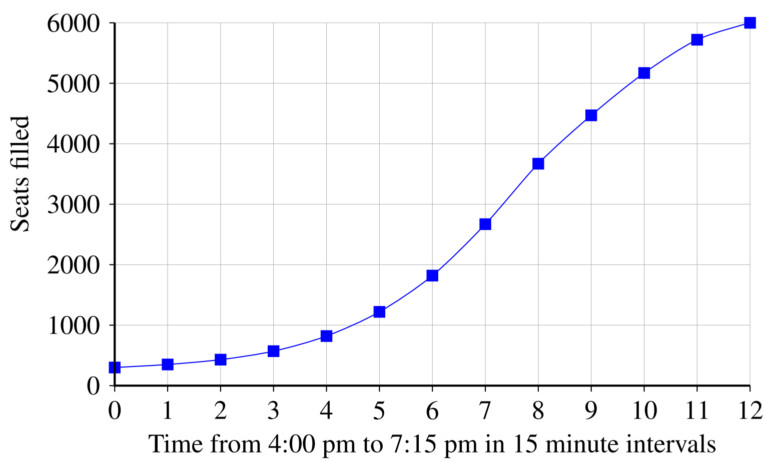

Figure 7A. Occupancy at an Event

This is a very good graphic: it tells us a lot about likely arrival times at the concert, observations we might conjecture happen at any concert. At the beginning the audience flow is slow, but around 4:20 the rate picks up and stays strong until about 6:40, when people begin to settled own. So, the model created by the smooth connection is good for suggesting tendencies, but it is not an accurate portrayal of the actual event. Since people are let in at 15 minute intervals, the audience count remains constant during those intervals, and jumps to a new count at the 15 minute markers. So, the audience is a constant 300 from 4:00 to 4:15, and a constant 820 from 5:00 to 5:15 at which time it jumps by 400, and so on, The largest jump is a t 6:00, when 1000 are let in all at once. Thus, the accurate graph of population count looks something like this:

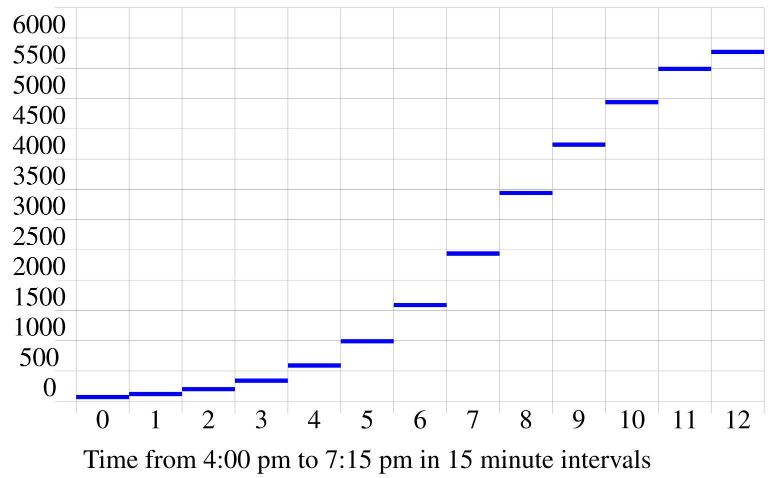

Figure 7B. Occupancy at an Event

When we connect data points with a "smooth" curve we say that we are creating a *continuous* model of the process. In many cases, this is much more useful than the "actual" graph (which is *discontinuous*) For example if we kept an audience count of a football stadium that seats 100,000 people, the count is discontinuous (it takes time for one person to get through the turnstile, and furthermore the count is always a positive integer), but that level of detail is not of interest and in fact would be a distraction. Similarly, stock market purchases are not continuous, but they happen at such a large rate that anyone studying stock market behavior would use a continuous model (as we see in the business section of the newspaper).

Sometimes it is important to distinguish *discrete* processes (those that change at separated instances) from continuous processes (like water flowing down a river). For example, consider the use of electricity in a typical home. We use electricity at a constant rate between the times an appliance is turned on or off. At those instances, the rate changes abruptly. Since flicking a switch is an assault on the system (too many working appliances could throw a breaker). These instantaneous changes are significant features of the study of electricity use; especially when considering the rate of use for a whole city. Use of air conditioners on a very hot day could, and sometimes do, shut the whole system. City planners want to understand this phenomenon so that they can plan for its occurrence. Hence in this context it makes the most sense to look at the discontinuous graph joining the data points, while, in the context of filling a football stadium, the continuous curve tells us more.

Before proceeding, this is a good place to introduce a little terminology that goes along with the function concept. First, the set of values for which the function is defined is called the *domain* of the function. The *range* of the function is the set of values that can appear as the output of the function. Let's look over the preceding examples to identify the domain and the range of the function.

- Example 1. If a table of corresponding (x, y) values is given, then the set of x values is the domain, and the set of y values is the range. In Example 1, the domain of the function is the set of values in the first row, and the range is the set of values in the second row. Figure 2 is a graph of that function. We went on to note (from either the table or the graph) that the difference between the y value and the x value is $3:15$ (using time notation). So, the equation $y - x = 3:15$ holds for all corresponding values of the function. Then we went on to model the given data by that equation, which defines a new function, whose graph is the straight line through the set of values of the first function. This new function is defined by the rule $y = x + 3:15$, whose domain and range are both the set of all possible times of the day. This gave us a way of predicting arrival times for buses leaving Salt Lake City at any time.
- Example 2. Here the function is defined by the rule: $y = 3x + 7$. To graph the function, we pick a set of values of x, and calculate the corresponding value of y, thus creating a table. Then we plot the points on a grid, and join these points with a line. It is important to stress that in this case the line *is* the graph of the function; we created a table by picking values of x arbitrarily, and then following the rules of the function. In fact, since we recognize the defining equation as that of a line, we need only have picked two values for

x. Since the domain of a functions is the set of values to which we can apply the rules, the domain of this function consists of all numbers on the number line.

- Example 3. In this example, we have already specified the domain: $x > 0$. Since the multiplicative inverse of a positive number is positive, the range also is the set of positive numbers. We might ask, what about negative numbers? Yes, the equation $y = 1/x$ does allow for a negative x, and for such an x the output will be negative. So, potentially, the equation $y = 1/x$ defines a function whose domain (and range) consists of all nonzero numbers. However, the proposer of the function is the one who gets to decide on the domain. Should she say that we want to only consider odd whole numbers divisible by 17, then that is the domain. But, in this case the proposer specified that the domain consists of all numbers greater than zero, and we conclude that this is also the range.

- Example 4. This is interesting because each time we make an improvement on the graph, we change the domain and range of the function. To begin with, the table of values is the set of all times at 15 minute intervals between 4:00 and 7:00, with the first data point recorded as 0 and the last as 12.

x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	300	350	430	570	820	1220	1820	2670	3670	4470	5170	5720	6000

The range is the set of positive integers on the second row of the table. These points are graphed in Figure 6. But now, in Figure 7, we've modeled the data with a smooth curve. Here, the domain is the set of *all* times between 4:00 and 7:00, and the range, the set of all numbers between 300 and 6000. Our last observation was that it doesn't make sense to say that there are 1035.76 people in the stadium at 5:08; In fact there are 820 and that has been true since 5:00 and will be true until 5:15. So, we drew the more accurate graph, Figure 8, of a function whose domain is all times between 4:00 and 7:00, but whose range has returned the set of positive integers in the second row of the table.

Functions Defined by Graphs

A graph provides a procedure for defining a function as follows:

For a value for x, draw the vertical line through that value. Where it hits the graph, draw the horizontal to the y-axis. That point is the value of y corresponding to the given value of x.

For this rule to work, we must know two things:

a. for a given number a, the vertical line $x = a$ intersects the graph;

b. for a given number a, the vertical line $x = a$ intersects the graph only once.

If these two conditions are satisfied, then the rule works: the vertical line through a (on the x-axis) intersects the graph at one point, and the horizontal line through that point intersects the y axis at some point b. This b is the value of the function for the input a. Keep in mind that the values for the function are approximate because a graph is a real object, not an ideal one. The precision of the approximation depends upon the detail in gridlines for the graph.

If either condition fails for a number a, then the function cannot be defined at a. We express this by saying that a is not in the *domain* of the function. Just to say this another way, a graph defines a function for all numbers a for which conditions **a** and **b** hold. That set of numbers is the *domain* of the function, and for any a in the domain, the above rule produces the value of the function at a. In some cases, if condition **b** fails: that is, the vertical line through some points on the x axis intersects the graph in more than one point, it may be possible to add a rule to the definition of the function that picks the correct point on the graph. For example, consider the graph of $y^2 = x$. For negative values of x, there is no intersection point of the graph with the vertical line, so no negative number is in the domain of the function. $x = 0$ produces only $y = 0$, so 0 is in the domain of the function. However, if $x > 0$, the vertical line intersects the graph in the positive and negative square roots of x. If we add the rule "y is not negative," then we have described a function for all non-negative numbers x: y is the non-negative square root of x. This function is denoted $y = \sqrt{x}$. We will review this in Example 9.

Example 5.

Create a data table for points on the graph in Figure 9.

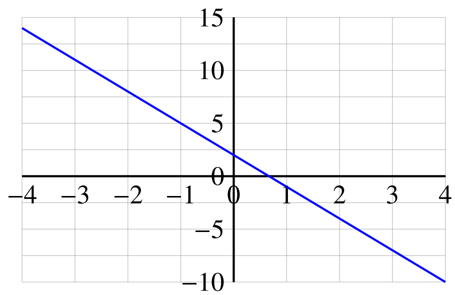

Figure 9

Applying the rule, we create this table of values of the function, by selecting integer values of x and making "best guesses" for the corresponding value of y on the graph.

x	-3	-2	-1	0	1	2	3
y	11	8	5	2	-1	-4	-7

We have selected integer values for x for convenience; however, given the statement of the problem, we could have selected any values. Also note that the y value of 11 corresponding to $x = -3$ was an estimate; 11.5 would have been as good (except that it creates more complicated arithmetic). It is a good rule to not try to be more accurate than the accuracy of the grid.

Observe that the graph is a straight line. We can pick any two points to find the slope of the line. Let's choose $(-2, 8)$ and $(1, -1)$ and calculate the slope:

$$m = \frac{8 - (-1)}{-2 - 1} = \frac{9}{-3} = -3$$

Since the y-intercept is given by the point $(0, 2)$, we know that $b = -2$, so the equation for this graph is $y = -3x + 2$.

Example 6.

Create a data table for points on the graph in Figure 10 (next page).

x	-1	-.6	-.3	0	.2	.5	.8
y	0	.8	.95	1	.98	.87	.6

In Example 5, the graph appeared to go through integer points so the pairs (x, y) were easy to find according to the set of rules for graphs. In this example we have to pick values of x for which we could most easily estimate values of y.

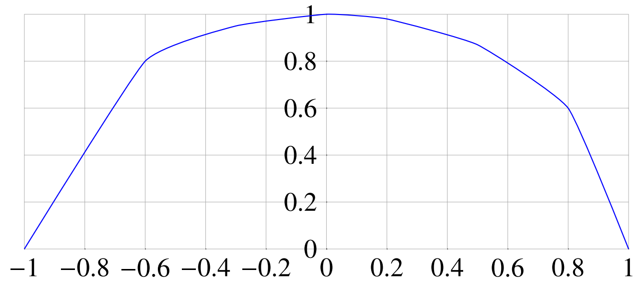

Figure 10

EXAMPLE 7.

Federico and Nkutete are hired at the same time by the Boston Envelope Company. However, they have different compensation contracts. Federico will start at an annual salary of \$25,000, with guaranteed raises of 6% each year, while Nkutete starts at an annual salary of \$20,000, with guaranteed annual increments of \$1,000. Which contract is better?

SOLUTION. The answer will depend upon how long they intend to work there. At the end of the first year, each gets a \$1,000 raise, so, Federico still earns more than Nkutete. The next year, Federico will get a slightly higher raise, but still has a lower salary. In fact, let's tabulate the effect on the salary of the annual raises for the first 12 years.

Year	0	1	2	3	4	5	6	7	8	9	10	11	12
Federico	25000	26000	27000	28000	29000	30000	31000	32000	33000	34000	35000	36000	37000
Nkutete	20000	21200	22472	23820	25250	26765	28370	30073	31877	33790	35817	37966	40244

By the twelfth year, Federico will have a higher annual salary, but, his cumulative income of \$318,343 is about \$24,000 less than Nkutete's cumulative income of \$342,000. Besides, it is altogether likely that they will both get promoted within the first ten years, so, the reasonable response to the question is that Nkutete is getting the better deal. Let us look at the graph of the data (Figure 11)

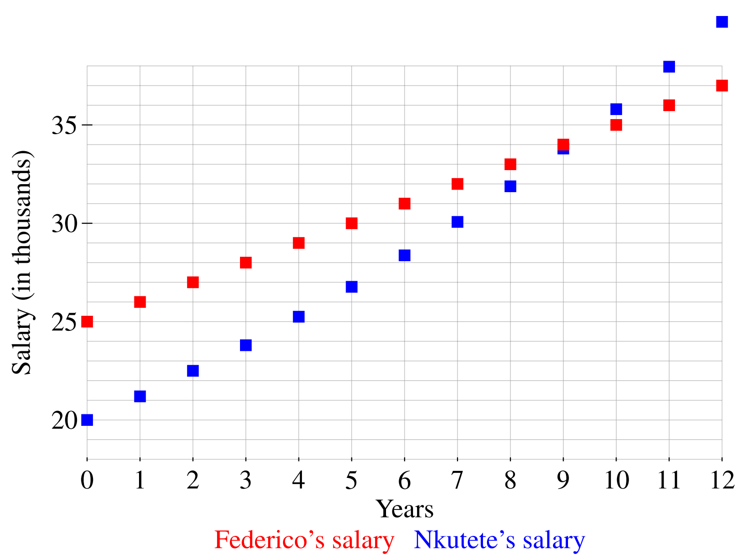

Figure 11

The graph clearly shows that, although Federico starts out with a higher salary, the gap between the salary decreases over the year, and ultimately, Nkutete is the higher earner. But what if they both worked at this same job for 40 years? In that span, who has gotten the better deal and by how much?

We will return to this example in the next section, where we will model these two graphs with curves that make the situation easier to understand.

Section 5.2: Linear and Nonlinear Functions

Interpret the equation $y = mx + b$ as a linear function. Observe that if b is not zero, the variables are not in proportion; however, the change is in the variables between two points are in proportion (hence the idea of slope). 8.F.3

Distinguish between linear and nonlinear functions.

The characteristic of a proportional relationship is that the quotient y/x, for values in the proportion, is always the same, and we call this the unit rate of y with respect to x. The significant characteristic of a line is this: for any two points P and Q, the ratio of the change in y from P to Q to the change in x from P to Q is a constant, called the rate of change of y with respect to x. This is an important characteristic: the variables in a linear relation are in proportion *only* when the graph of the relation goes through the point $(0, 0)$. To see this algebraically: if y is a linear function of x; that is $y = mx + b$, the the quotient y/x, for $x \neq 0$ is

$$\frac{y}{x} = m + \frac{b}{x}$$

which is definitely not constant for $b \neq 0$.

Important things to remember about linear functions are:

- If the line intersects the y-axis in the point $(0, b)$, then the equation of the line is $y = mx + b$.
- If the line is horizontal, the slope is zero, and the equation of the line is $y = b$.
- If the line is vertical, it has no slope, and its equation is $x = a$.
- If the line goes through the origin, the equation of the line is $y = mx$ and the values of y are proportional to the values of x; otherwise said, $y/x = m$.
- If the line has slope m, and the point (x_0, y_0) is on the line, then the equation of the line is

$$y - y_0 = m(x - x_0)$$

- If (x_0, y_0), (x_1, y_1), then a point (x, y) is on the line if

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

In this section, we look at a variety of examples of functions in various representations (formula, table, graph) to make clear the distinction between linear and nonlinear functions. First, llet's go back to the hires of the Boston Envelope Company (Example 7). The data in Figure 11 shows the gap in salaries decreasing, but it doesn't make clear what the relationship will be in the long run. We can make it more clear by connecting the points with curves that are as simple as possible (see Figure 12).

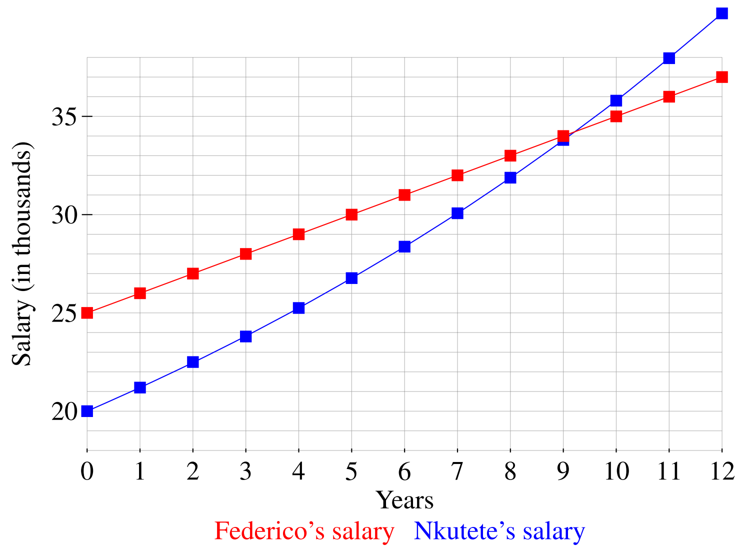

Figure 12

The curve modeling Federico's salary is a straight line, while that for Nkutete is not; neither the table nor the graph of data points showed a tendency for Nkutete's salary curve to become steeper and steeper. If we calculate with the table, we can see that the rate of change of Nkutete's salary gets larger over time; the value of the graph is that it shows us this instantly.

As we continue to study data for two variables, looking for a relation between them, we hope to find a formula that actually exhibits one variable as a function of the other. This will allow for prediction of future pairs of values not on our table. To set the ground for this, we look at a collection of examples represented in various ways: formula, tables or graphs.

Example 8.

$y - 2x = 11$

When we start, as in this case, with a formula relating two variables, it is not clear which is a function of the other, if at all. Often the context tells us what choice to make, other times it is desirable for reasons of computation, to make a choice. In this case, we could write y as a function of x:

$$y = 2x + 11$$

or x as a function of y:

$$x = \frac{y - 11}{2}$$

Since the first is simpler, let's make that choice, and create a table like this:

x	-4	-3	-2	-1	0	1	2	3	4
y	3	5	7	9	11	13	15	17	19

Notice that every time x increases by 1, y increases by 2. Recall that this tells us that 2 is the slope of the line, or the unit rate of change. See the graph (Figure 13, next page).

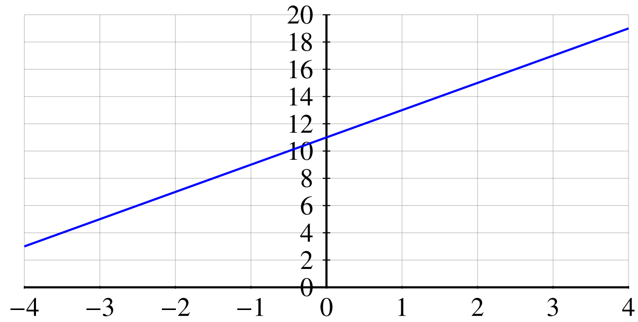

Figure 13

Extension

Example 9.

$y^2 = -x + 1$

Here, it is easier to make the table by writing the relation in the form $x = 1 - y^2$ and finding values of x corresponding to values of y. This gives us the table and graph

x	-8	-3	0	1	0	-3	-8
y	-3	-2	-1	0	1	2	3

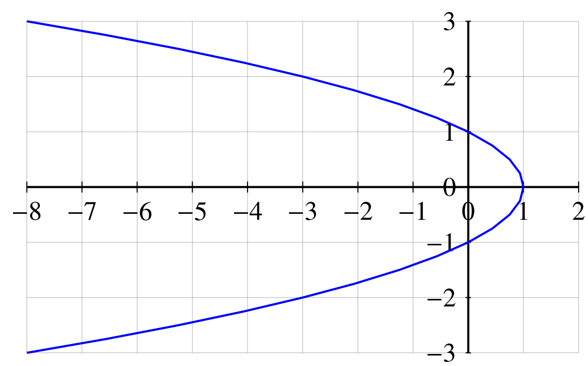

Figure 14a

We see from this graph that this does not specify y as a function of x; at least not until we include a rule that tells us, for any x which of the two candidate values is to be chosen. We may, for example, add the rule: For each x, let y be the positive number such that $y^2 = -x + 1$. Then we get this graph, which now describes a function:

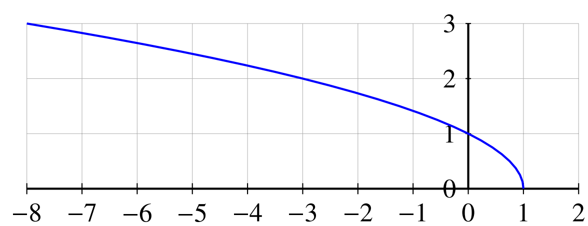

Figure 14b

To summarize: figure 14a (of the relation $y^2 = -x + 1$) does not describe y as a function of x, because of the ambiguity in taking the positive or the negative solution of $y^2 = -x + 1$. This ambiguity is resolved by adding the stipulation: For a given $x < 1$, let y be the positive solution of the equation $y^2 = -x + 1$, resulting in figure 14b.

End Extension

Section 5.3: Modeling and Analyzing a Functional Relationship

This section - like many of the topics in 8th grade - is exploratory, with the goal of understanding functional relationships in context. There are two processes to be introduced and explored. First, suppose that we are considering two variables (for example: the height and girth of a maple tree) that we think might be functionally related. We gather data on the variables, specifically pairs of the values of the two variables in a sample set, or in experiments. Finally, we study the variables in a variety of ways to see if we can find a model (a specific set of rules defining a function) that fits the data well enough to be able to make predictions on the outcomes of further sampling or experimentation. Second, we may be given a functional relationship, that is, a set of rules that determine values of the second variable dependent upon values of the first. This may take the form of an equation involving the two variables, or an algorithm to compute one from the other. In this case we study the functional relationship in a variety of representations (tables, graphs and equations) to see if we can understand the properties of the relationship. In this chapter we shall look at what mathematicians call the *deterministic* data; for example two different ways of measuring the same physical attribute. In the next chapter, we explore how to (best) do this with data gathered at random, and thus subject to random inputs. In our first few examples, the context clearly indicates a constant rate of change, and thus, a linear relationship. The subsequent examples show a variable rate of change; here we explore what we can learn from the data. In every case, we make a choice of one of the variables, x as the variable upon which the other variable, y is dependent. We determine, from the graph, in what intervals y is increasing/decreasing as x increases, and we begin to understand the significance of those points where this behavior changes.

Constructing Functions

Construct a function to model a linear relationship. Determine rate of change, initial value (use representations and context). 8.F.4

Example 10. Heat and Temperature

The temperature of an object measures the amount of heat it contains. Temperature is measured in degrees, denoted °. No matter what scale is used, it should be such that a change in the temperature of an object is proportional to the change in heat content expressed in some other measure, such as calories. However, caloric content is hard to measure directly, and so we turn to other means to quantify heat. For example, some fluids expand in volume as they heat up, and in a linear way: the change in volume is proportional to the change in caloric content. Mercury is such a fluid, and thus is the fluid of choice in a thermometer. As the object heats up, the mercury expands and the column of fluid in the stem of the thermometer rises. The important thing is that this change in height of the column of mercury is proportional to the change in volume, and thus proportional to the change in heat.

The Celsius temperature scale is based on water: 0 °C corresponds to the heat content of newly melted ice, and 100°C to water just starting to boil. Thus, if a pot of water measures 50°C, the increase in caloric content from 0°C is half the increase in caloric content of the same amount of water at the boiling point. Daniel Gabriel Fahrenheit was a doctor in the 18th century who wanted to measure the heat generated by a disease in a human patient, so he invented a scale that was based on humans: 100 °F is the temperature of a healthy human being (approximately), and 0°F is the temperature of blood just about to freeze. By experimentation, Fahrenheit related his sclae to the Celsius (at that time called "Centigrade) scale by experimentation: finding that the freezing point of pure water is 32°F, and the boiling point is 212 °F.

Given that changes in these two temperature scales are proportional to caloric content, they must have a constant rate of change with respect to each other. Thus we can relate °C with °F by a linear relation. We know two points on the graph of this relation: the freezing point of water, $(0, 32)$ and the boiling point of water, $(100, 212)$ (where we have put °C as the first coordinate). The slope of the line graphing this relation is

$$\frac{212-32}{100-0} = \frac{9}{5}$$

This can be stated this way: a 9 degree change Fahrenheit is the same as a 5 degree change Celsius. As a ratio: *change in °F: change in °C*::9:5. We also know the y-intercept: it is 32, since (0,32) is on the graph. Thus the function relating Fahrenheit to Celsius is

$$F = \frac{9}{5}C + 32$$

Now, we can express Celsius as a function of Fahrenheit, by solving for C in terms of F:

$$C = \frac{5}{9}(F - 32)$$

Example 11.

At Mario's Cut Rate Used Car Lot, Mario compensates his salespeople with salary plus commission: each salesperson receives a base salary and then a certain amount for each car sold. His more experienced people get a higher base salary, but the new people get a higher commission, because he wants to encourage them to be eager to sell cars. Sally, his most seasoned salesperson receives a salary of $4,000 per month and a commission of $250 per car sold. Dmitri is a rookie, receiving a salary of $2,800 per month, but his commission is $325 per car sold. Let's see what we can learn from examining these two means of compensation.

Here is a table of Sally's and Dmitri's earnings at 0,1,2,3,...,8 cars sold.

Cars Sold	0	1	2	3	4	5	6	7	8
Sally	40	42.5	45	47.5	50	52.5	55.0	57.5	60
Dmitri	28	31.25	34.5	37.75	41	44.25	47.5	50.75	54

If we were to graph the data, we'd get two straight lines since, for each salary the rate of change with respect to number sold is constant. Since the rates (250 and 325) are not the same, the graphs intersect. The point of intersection tells us the number of cars each must sell in order to have the same salary. From the graph, we can read off the coordinates of that point, or we can do that algebraically: Let N represent a number of cars sold, and S, Sally's compensation for N cars sold, and D, Dmitri's compensation for N cars sold. The rules can be written algebraically as

$$S = 4000 + 250N \quad , \quad D = 2800 + 325N$$

At the point of intersection the two salaries are the same ($S = D$,) so we have to find out, for what N is $4000 + 250N = 2800 + 325N$? The solution of that equation is $N = 16$. At 16 cars sold, Sally and Dmitri receive the same compensation, $8,000.

Example 12.

A bookseller is trying to set a price for her books in such a way as to keep the carry-over inventory at an acceptable level. She decides to vary her prices, month by month for a little over a year, to see the relationship between price and inventory. Here are the data:

Month	Oct	Nov	Dec	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep
Price	1.4	1.10	1.00	1.4	1.80	2.40	1.60	2.00	2.50	3.50	2.65	1.50
Inventory	90	98	75	55	98	146	115	100	110	175	125	92

We can't tell much from these data, except that each time the price was lowered, the unsold inventory was lowered. Maybe, if we reorder according to price, with the inventory as the dependent variable, we get a different picture. In fact the picture we get is this:

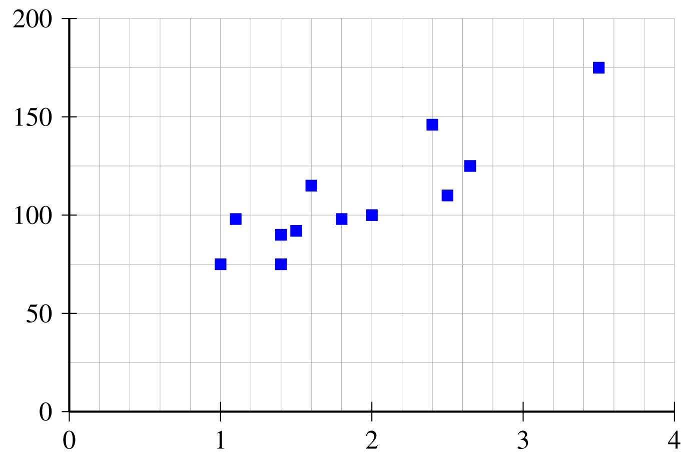

Figure 15

We see several things that are not readily apparent from the table: generally speaking, as the price rises, so does the inventory. We could conclude that, if we never want an unsold inventory of more than 150 items, then we should keep the price under $2 (well, almost all the time, in one month out of 10 this was not true). We also don't discern any curving of the data, so we might surmise that the relation (except for random variations) is linear.

In general, if we are given a table of data, we should first determine (from the context) which variable should be the horizontal, and which, the vertical. Then we should reorder the table in increasing order in x. We can now check for linearity: if the change in y is proportional to the change in x (that is, given any two points, the quotient of these changes is always the same number), then the data are that of a linear relation. An easier way, and one which in any case gives good information, is to plot the points to see whether or not they lie on a line. The data may have come from measurements which are prone to random error. If the points almost lie on a line, but do not actually lie on a line, we may be able to conclude that the relation is linear.

Extension. Analyzing a Functional Relationship

Data that are collected from real contexts, such as a laboratory experiment, or a questionnaire, are very unlikely to fall on a line - or for that matter in any precise pattern. Nevertheless, the graphed data may show a trend or suggest a relation, or uncover an anomaly.

Example 13.

On average over the past 175 years, the hour by hour temperature, from 1 am to 9 pm, for an August day in Salt Lake City is:

Time of day:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Temperature:	73	72	71	70	69	69	70	72	76	81	74	84	85	87	89	90	89	88	68	85	84

Figure 16 is a plot of the points on a graph. We have connected those points with a smooth curve:

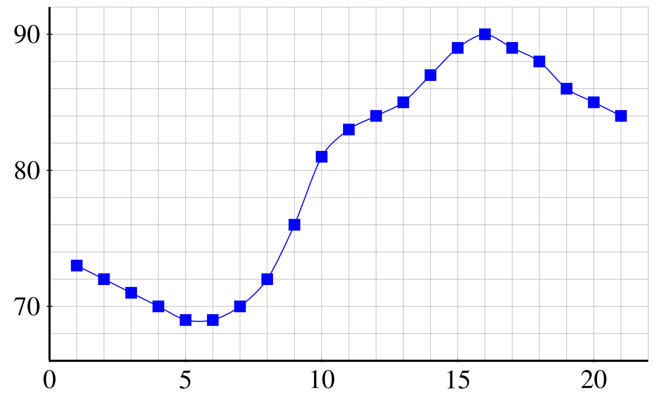

Figure 16: Average August Temperature

The graph confirms some things that we should have expected on physical grounds: that the temperature rises during the day as the sun moves directly overhead, and drops - more or less linearly - when the sun is down. We also see that the highest temperature is later in the day than we might have suspected; suggesting a cooling-off lag.

Now let's look at the data for a particular day: August 26, 2012:

Time of day:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Temperature:	78	76	75	74	73	72	73	75	77	82	86	90	92	94	84	78	82	84	82	81	80

Here is the graph:

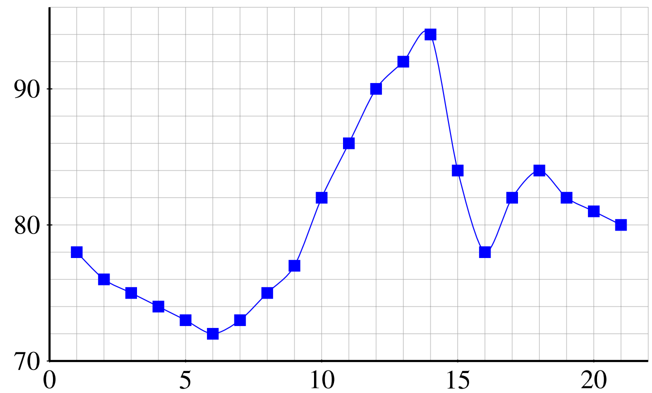

Figure 17:Temperature, Aug. 26,2012

This starts out like a typical August day, but then there is a sudden decrease in temperature at 15:00 for about 2 hours. This suggests a thunderstorm or the arrival of a cold front. However, the data show a warming up around 17:00, returning to the "typical day." This is not what happens when a cold front arrives, so strengthens the argument for a thunderstorm.

Let's look back at Figure 16, of average temperatures for August. Notice that the temperature rises at a steady state until about 3 pm where it flattens out a bit. Since this is the average temperature, this blip suggests that afternoon thunderstorms occur in August frequently enough to affect the average. This example is important because it illustrates the difference between "average" and typical. Figure 16 is not the temperature picture for a typical day, for there are (at least) two typical days, one with no afternoon thunderstorm (in which case the temperate will rise steadily until about 5 pm), and the other typical day with a thunderstorm.

End Extension

Extension. Eventually, we will discover that there are just a few most important graphs that come up while trying to model situations. The following set of problems ask this: given a graph, can one describe a process, or situation that is modeled by this graph?

EXAMPLE 14.

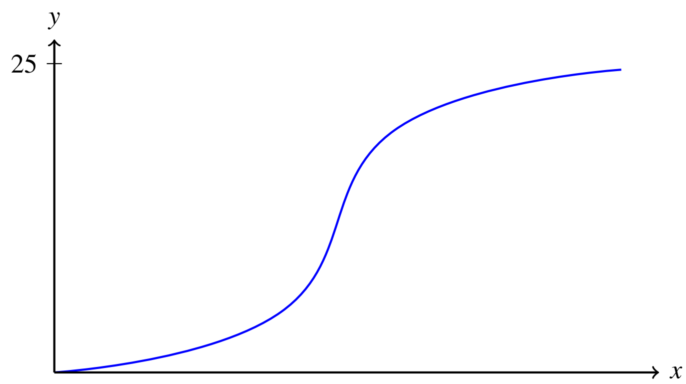

Figure 18

The curve in Figure 18 starts at $(0, 0)$. As x increases, at first y increases slowly, then during the next period increases rather rapidly, and finally levels off at about $y = 25$. If you have ever gone to a concert in a basketball arena, you have seen this behavior. In this model, time is on the x axis, and the y axis shows the number of people in the arena (in thousands). At first, people dribble in slowly, and the rate at which people enter the stadium rises rapidly, then rapidly lowers. Just as the game starts, the remaining seats get filled more and more slowly.

EXAMPLE 15.

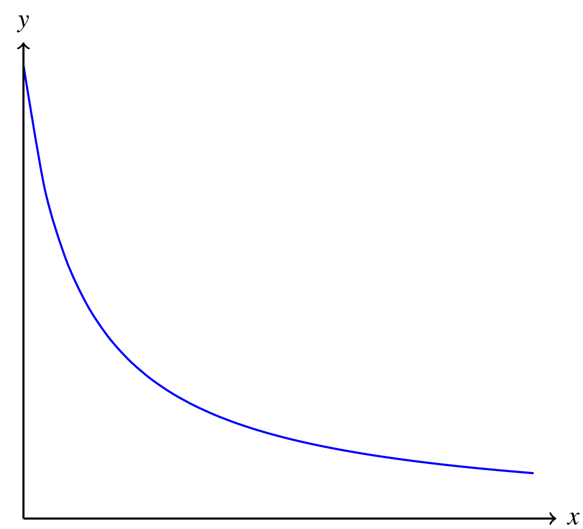

Figure 19

Figure 19 looks like the profile of a ski run, suggesting that it is a graph of altitude against time as I ski down a black diamond run that levels out at the bottom.

EXAMPLE 16.

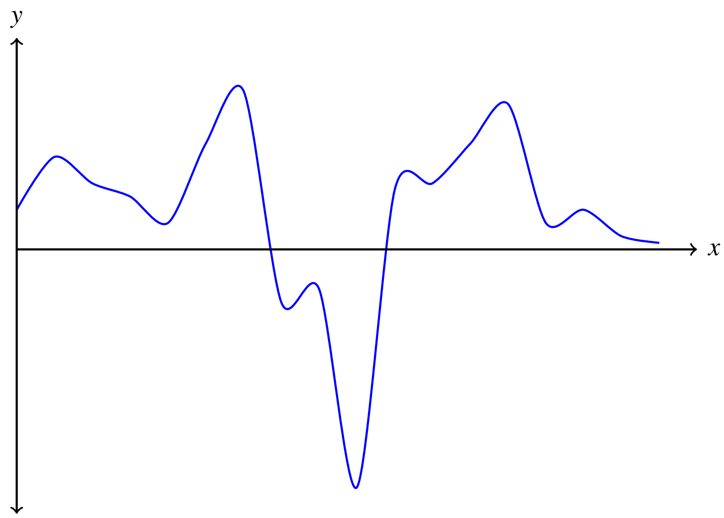

Figure 20

This looks like a random graph, but our task is to find a context which might lead to this graph. One thing that comes to mind, is that these are two islands separated by a deep trench. The x-axis can be interpreted as sea level. In particular, x is distance along a straight line cutting through both islands, and y is the altitude above (or below) sea level.