Chapter 1
Linear Equations in One Variable

The first three chapters of grade 8 form a unit that completes the discussion of linear equations started in 6th grade, and their solution by graphical and algebraic techniques. The emphasis during these three chapters moves gradually from that of "equations" and "unknowns" to that of "functions" and "variables." The first chapter is about solving linear equations; in the second we move to the graphical interpretation of linear expressions and the understanding of the constant rate of change for linear functions (to be compared to the constant of proportionality of a proportional relation), and then to the "slope" of a line, and how it can be interpreted as the rate of change of the associated linear expression. Note that in this discussion we have been using the word *expression* rather than *function*, so that the students will become familiar with the idea of evaluating linear expressions, and graphing those values, as a lead in to the concept of function. This is engaged in the third chapter in which the entire subject of linear functions is brought together and examined from a variety of perspectives. In Chaprer 4 we return to solutions of equations, this time with pairs of equations in two variables for which we seek values of the variables that solve both equations (the solutions of the simultaneous equations). Finally, in Chapter 5 our focus turns completely to the concept of function, shifting emphasis to describing the relation between the two variables, rather than the mechanics of the function.

This chapter begins with a focus on the distinction between expressions and equations. The analogy is with language: the analog of "sentence" is *equation* and that of "phrase" is *expression*. An equation is a specific kind of sentence: it expresses the equality between two expressions. These equations involve certain specific numbers and letters. We refer to the letters as *unknowns* - that is they represent actual numbers which are not yet made specific; indeed, the task is to find the values of the unknowns that make the equation true. If an equation is true for all possible numerical values of the unknowns (such as $x + x = 2x$), then the equation is said to be an *equivalence*. Arithmetic operations transform expressions into equivalent expressions; we come to understand that for linear expressions, the converse is true: we can get from one expression to an equivalent one by a sequence of arithmetic operations.

In chapter 2 we begin to change the way we look at the letters used in algebra from that of *unknown* to *variable*, and together with that, the understanding of an equation involving two variables as expressing a relation between them. We do this first in the context of proportion, but then go to general linear equations and the ideas of *rate of change* and *slope* of the graph. But in chapter 1 we are only interested in finding that (or those) number(s), if any, which when substituted for the unknown make the equation true. These are the *solutions*. We do this in specific contexts, seeing how to translate language sentences about numbers into equations involving linear expressions.

Linear Expressions

A *linear expression* is a formula consisting of a sum of terms of the form ax and b, where a and b are numbers and x represents an unknown. By *unknown* we mean a symbol which stands for a number; it could be a specific one yet to be determined, or one yet to be chosen, or any possible number, depending upon the context. Since "x" represents an unknown, we could replace it by any letter and still have the same sentence. In particular, in solving a problem in context, it is a good idea to pick a letter suggested by the context.

Example 1.

These are all linear expressions:

a. $3x - 5$ **b.** $3t$ **c.** -5 **d.** $3u + 2u + 17$ **e.** $3x - \frac{12x - 16}{4}$ **f.** $\frac{2}{5}y + \frac{3}{10}$ **g.** $6(2x - 5) + 11$

Notice (as in examples **b.** and **c.**) that we could have either a or b, or both, equal to zero. When we have a particular problem, we are unlikely to start with a linear expression of the form **c**, but, in our manipulations with the expression, we may end up there. For example, when we combine terms in **e** we end up with $3x - 3x + 4$, which is just 4.

To *evaluate* a linear expression is to substitute a number for the unknown and calculate the resulting value. For example, if we evaluate **d** at $u = 1$, we get 22, at $u = -3.5$, we get -0.5. Often we are interested in how the value of the expression changes as we change the value of x and evaluate, and so we generate a table of corresponding values of x and the expression, and go on to graph those points on a coordinate plane. Were we to do so, it always turns out that the points lie on a straight line, and that is why expressions of this form are called "linear". But we are getting ahead of ourselves; we will return to this discussion in the next chapter.

It is often the case that different linear expressions have the same meaning: for example $x + x$ and $2x$ have the same meaning, as do $x - x$ and 0. By the *same meaning* we mean that a substitution of any number for the unknown x in each expression produces the same numerical result.

Example 2.

Consider these linear expressions:

a. $2(x + 5)$ **b.** $2x + 10$ **c.** $x + 10$ **d.** $2x + 5$

a. and **b.** have the same meaning, since every substitution of a number for x gives the same result. As for **a.** and **c.** , although the substitution of 0 for x produces the same result, it will not work for any other number. And the situation for **a.** and **d.** is even worse: there is no number for which they both give the same result,

How do we know that **a.** and **b.** have the same meaning, since we cannot test *every* number? The answer lies in the laws of arithmetic. We say that two linear expressions are *equivalent* if we can move from one expression to the other using the laws of arithmetic. When two linear expressions are equivalent, they have the same meaning: to be precise, any number substituted for the unknown in the expression always returns the same value. Indeed, this is why the "laws of arithmetic" are called laws: they preserve the meaning of the expression. This is very convenient: to show that two expressions have the same meaning, we do not have to check every number (an impossible task in any event); it is enough to show that we can move from one expression to the other using the laws of arithmetic. On the other hand, to show that two expressions are not equivalent, just find one number that gives different results when substituted for the unknown in the expressions.

In this chapter we want to work precisely with these ideas of equivalence of linear expression, leading to *simplifying* and *solving*. In the next chapter we will see that the graph of a linear expression (where y is the *output* of the expression for the *input* x) is a straight line. Since a line is determined by any two points on the line, we will see that to test the equivalence of linear expressions it suffices to test them for just two values of the input x.

Any linear expression is equivalent to one in the form $ax+b$ using the laws of arithmetic. This process is sometimes called *simplifying* the expression. However, *simplify* is not always what we want to do with an expression. For example, if we change $7x - 28$ to $7(x - 4)$, then we learn that in the equation $y = 7x - 28$, y is proportional to $x - 4$. So, if we are interested in the behavior of x and y in the equation $y = 7x - 28$, the form $y = 7(x - 4)$ is "simpler." In general, the word *simplify* should be tied to the goal of the work being done; right here it is to minimize the number of symbols necessary to understand the expression. Let's go through Example 1 to simplify each expression, using arithmetic operations, into the form $ax + b$ for some numbers a and b. As for examples **a**, **b**, **c** and **f**, they are already in that form. Let's now look at the others, and a few more just to cover all the possibilities.

Example 3.

a. example 1d: $3u + 2u + 17$ is equivalent to $5u + 17$. For we can combine similar terms: "three u's plus two u's" is the same as "$5u$."

b. example 1e: By doing the division implied by the fraction, we see that $3x - \frac{12x-16}{4}$ is equivalent to $3x - 3x + 4$, which is just 4 (as mentioned above).

c. example 1g: Distribute the 6 in $6(2x - 5) + 11$ to get $12x - 30 + 11$, and now add $-30 + 11$, to get $12x - 19$.

d. $4x + 5$ and $2x + 3x + 5$ are not equivalent. Substitute 1 for x, and obtain 9 in the first expression and 10 in the second.

e. $3x + 5$ and $6x - 1$ are not equivalent: if we substitute 1 for x, we get 8 in the first expression and 5 in the second. But be careful: if we substitute 2 for x, we get the same result: $3(2) + 5 = 11$ and $6(2) - 1 = 11$. Since we can find at least one value for the unknown that gives different results to the expression, they are not equivalent.

Example 4.

$6x - 20 + 2(x - 4)$ and $4(2x - 7)$ are equivalent.

Let's go through the steps, giving clear reference to the relevant laws of arithmetic.

Step 1. Start with $6x - 20 + 2(x - 4)$. Distribute the 2 to remove the parentheses, to get: $6x - 20 + 2x - 8$.

Step 2. Combine like terms to get: $8x - 28$

Step 3. Factor out 4 to get $4(2x - 7)$ which is precisely the second expression.

There are many ways to go from one expression to an equivalent one. For example, we could distribute and collect terms in both expressions to obtain $8x - 28$ from each. To put this another way: two expressions are equivalent if they are both equivalent to a third expression.

In summary, the end result of simplification is an expression of the form $ax + b$: This is always the case: any linear expression simplifies to the form $ax + b$. The word "simplify" is often ambiguous - it usually depends upon where it is you want to go with the expression; in this case, it is to the form $ax + b$.

Section 1.1: Solving linear equations: obtaining the desired value of an expression

Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms. 8.EE.7.ab

In the introduction to this chapter, we talked about "evaluating expressions". Here we ask: given a linear expression, and a number c, for what value of the unknown does the expression compute to c? This can be restated as: given the expression $ax + b$ and a number c, find the value of x that produces that c. Let's first review what was done in grade 7.

Example 5.

a. For what x does $2x + 5$ evaluate to 17? Otherwise put: solve $2x + 5 = 17$.

Solution. Subtract 5 from both sides of the equation to get $2x = 17 - 5$. Replace $17 - 5$ by 12 to get $2x = 12$. Now divide both sides by 2 to get $x = 6$.

b. For what x does $2(x + 5)$ evaluate to 24? Otherwise put: solve $2(x - 5) = 24$.

Solution. Divide both sides by 2 to get $x - 5 = 12$. Now add 5 to both sides: $x = 17$.

Note that in the second problem, we'd rather not use the distributive property: it is easier and quicker to first divide by 2 than to distribute the 2.

Now we want to work more complicated expressions. The procedure will be the same, except that first we have to appropriately simplify the expression. Let's work with the expressions **e**, **f** and **g** of example 1.

Example 6.

a. For what value or x is $3x - \frac{12x - 16}{4}$ equal to 5?

Solution. First, we reduce the fraction to obtain the equation $3x - (3x - 4) = 5$, and then use the distributive property to get $3x - 3x + 4 = 5$. Then combine terms to obtain $4 = 5$. Since 4 is not equal to 5, there is no value of x to obtain 5 from this expression.

b. Let's slightly change the expression so as to obtain a more satisfying result. For what value or x is $5x - \frac{12x - 16}{4}$ equal to 25?

Solution. Again, we reduce the fraction, this time obtaining $5x - 3x + 4 = 25$. Combining terms, this becomes $2x + 4 = 25$, which has the solution $x = \frac{21}{2}$.

Example 7.

For what value of x is $\frac{2}{5}x + \frac{3}{10}$ equal to 0.375?

Solution. Otherwise put, solve

$$\frac{2}{5}x + \frac{3}{10} = 0.375$$

First, multiply both sides by 10 to obtain

$$4x + 3 = 3.75$$

Subtract 3 from both sides getting $4x = 0.75$, and divide by 4 getting $x = 0.1875$

This is a good time to point out that there can be many ways to solve a problem, and in this case, there may be better ways. Noticing that the notation is hybrid (we have both fractions and decimals) we could move to one notation or the other.

Yet another way would be to write all numbers as decimals to get $0.4x + 0.3 = 0.375$ and now multiply by 10 to get $4x + 3 = 3.75$, and now proceed as above.

Write all numbers as fractions to get

$$\frac{2}{5}x + \frac{3}{10} = \frac{3}{8}$$

Multiply by 40 to eliminate denominators, getting

$$8(2x) + 4(3) = 5(3) \qquad \text{and then} \qquad 16x + 12 = 15$$

Subtract 12 from both sides to get $16x = 3$, and finally divide by 16: $x = \frac{3}{16}$

Example 8.

a. For what value of x is $6(2x - 5) + 11 = 53$?

b. Given a number y, for what value of x is $6(2x - 5) + 11 = y$?

Solution.

a. We want to illustrate two different ways to solve this problem.

a1. Distribute the 6 and add −30 to 11: $12x - 19 = 53$,
Add 19 to both sides: $12x = 72$, and now divide by 12 to get $x = 6$

a2. Subtract 11 from both sides : $6(2x - 5) = 42$,
Divide by 6: $2x - 5 = 7$
Add 5 to both sides: $2x = 12$,
and now divide by 2 to get the answer $x = 6$.

b. This example is here to illustrate that these processes of solving work no matter what is on the right side - even if it is the *unknown* y. Eventually we want to understand the situation more deeply. The given expression (in **b** of the problem) describes the procedure that takes us from a given x to the answer y. The question asks for the procedure that starts with a given value of y and produces the x that makes the equation true. So we want to write the equation in the form $x = \cdots$, where the dots is to be filled with an expression in y. We do this by simply applying the process used for part **a.** First, apply the first two steps of **a1** to get $12x - 19 = y$. Now add 19 to both sides to get $12x = y + 19$. Now divide by 12 to get the result

$$x = \frac{y+19}{12}$$

Section 1.2: Solving linear equations: equating two expressions

A *linear equation* is an assertion that two linear expressions are equal. In the above, we have considered the case where one of the expressions is simply a number, and put it in the context of evaluation of expressions. Now we want to find out for what value of the unknown two expressions produce the same result. This may seem more difficult to the students, but the ideas are precisely the same. The difficulty may be this: it is clear that we can subtract 5 from both sides of the equation to get an equivalent equation, but since we don't know what x is, is it really all right to subtract $5x$ from both sides? Of course it is, since x does represent a specific number, and so the laws of arithmetic apply. Later, when we move from the concept of "unknown" to that of "variable", then x is a quite different object, representing not some particular number that we don't know just yet, but any possible number. Nevertheless, the same reasoning applies: the laws of arithmetic actually do hold for any numbers and any expressions.

If a linear equation is an assertion that two linear expressions are equal, then "solving" the equation is to find out for what numbers the assertion is true. Two linear equations are equivalent if one can be obtained from the other by a succession of applications of laws of arithmetic. The goal of solving the equation is to find a sequence of equivalent equations, starting with the given equation and ending up with something like $x = 5$. Of course, we may not end up there: just as the expression $x - x + 1$ is equivalent to the expression 1, the equation $x - x + 1 = 2$ is equivalent to $1 = 2$, which of course is false. Since it is false, no matter what value x takes there is no solution.

In general, the result of this process may be "all numbers" or "a particular number" or "no numbers". Let's look at some examples:

Example 9.

a. $2(x - 5) = 3x - 1$ **b.** $2(x + 5) = 2x + 10$ **c.** $7 = 5$ **d.** $7x = 5x$
e. $3(x - 5) = 2x$ **f.** $7 = 5 + 2$ **g.** $7x = 7x + 1$

The examples presented here are designed to indicate the breadth of issues that may come up as students learn this subject, and not to provide instructions. As you look through them keep in mind that the truth or falsity of the equation is something to be determined: it is our task. This is different from the validity of the equation as a statement. To illustrate: "Julius Caesar was the first President of the United States" is a valid statement, but false. The assertion "Mr. XXX was the first President of the United States" is a valid statement, but doesn't tell us much (except that that person was male). The equally valid and true statement is "George Washington was the first President of the United States'." Analogously, in example 8, **a.** is true for one value of x, **b**. for all values of x and **c.** for no values of x. If the equation is true for the substitution of every number for the unknown, it is an equivalence. Now, the reader might conclude that we cannot substitute *every* number for the unknown, so we can never be sure it is an equivalence. However, for linear equations, after we verify in the next chapter that the graph is a straight line, it follows from the fact that a straight line is determined by just two points, that we need only check two values of x. Now, if an equation is not an equivalence, it still may be true for some substitutions of x (these are called the solutions), or there may be no substitution to make it a true statement.

There are various techniques for solving a linear equation; all techniques amount to applying arithmetic operations to the equation that do not change the set of solutions. There are three kinds of operations:

1. **Apply the laws of algebra to simplify the expressions**; in particular, distribute to remove parentheses and combine like terms.

   Transform the equation $2x + 3x = 5 + 20$ to the equation $5x = 25$. Transform the equation $6(x - 2) = 11$ to the equation $6x - 12 = 11$.

2. **Add or subtract the same expression to both sides of the equation.**

   Transform $3x = 2 - x$ to $4x = 2$ by adding x to both sides.

3. **Multiply or divide both sides of an equation by a nonzero number.**

   Transform the equation $2x = 8$ to $x = 4$ by dividing both sides of the equation by 2. Transform $3x = 6x - 18$ by first dividing by 3 to get $x = 2x - 6$, and then combine like terms to find $x = 6$.

These operations all transform any equation into another with the same set of solutions. What is most important is that they are effective: they succeed in solving any linear equation. Let's apply these ideas to the equations of parts **a** through **g** of example 9.

Example 9 Solutions.

a. $2(x + 5) = 3x - 1$;

Simplify the left side: $2x + 10 = 3x - 1$;

Subtract $2x$ from both sides: $10 = x - 1$;

Add 1 to both sides: $11 = x$. Thus there is one solution: $x = 11$.

b. $2(x + 5) = 2x + 10$;

Simplify the left side: $2x + 10 = 2x + 10$. Since both sides are the same expression this is true for all values of x; that is, the expression on both sides of the equals sign in b) are equivalent. Consequently, every number is a solution to this equation.

c. $7 = 5$ this is false: If we think of this as $7 + 0x = 5 + 0x$, we can assert that there is no value of x to make it true.

d. $7x = 5x$: Subtract $5x$ from both sides: $2x = 0$. Divide both sides by 2: $x = 0$, so 0 is the only solution.

e. $3(x - 5) = 2x$;

Simplify the left side: $3x - 10 = 2x$;

Subtract $2x$ from both sides: $x - 10 = 0$;

Add 10 to both sides: $x = 10$ Therefore, there is one solution: $x = 10$.

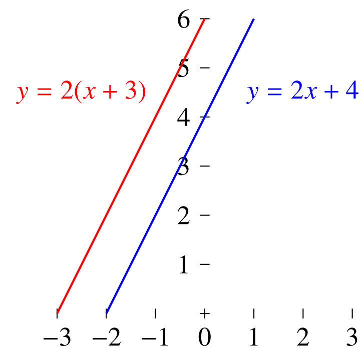
f. $7 = 5 + 2$: combine like terms on the right: $7 = 7$. This is a true statement.

g. $7x = 7x + 1$: Subtract $7x$ from both sides: $0 = 1$. This is a false statement, so there is no solution to the original statement: no choice of value for x will make it true.

Let us take a moment to notice the exceptional cases: **b**, **c** and **g**, where we do not get a solution, but either get an equivalence (all numbers solve the equation) or there is no solution. These occur when the coefficient of x is the same on both sides of the equation. We see this in **b** after distributing the 2, and in **c** and **g** the equation starts that way.

There is a geometric representation of this situation that helps make this clear. Suppose that we start with the expressions $2x + 4$ and $2(x + 3)$. Let's graph the two expressions: that is, graph the equations $y = 2x + 4$ and $y = 2(x + 3)$.

In general, when we graph two linear expressions, we get two lines, and the point of intersection gives the value of x for which the two expressions give the same number. However in this case (see the figure), the lines are parallel, so there is no point of intersection; explaining why there is no solution. Now, if we had started out with the expression $2x + 4 = 2(x + 2)$, then the figure would have shown just one line, since the two expressions are equivalent.



An important feature of the allowable operations on equations is that they can be reversed: if an operation takes one equation to another, it can be undone:, meaning that there is an operation on the second equation that produces the first.

Example 10.

Solve $-3x + 8 = 20 + x$.

Step 1: Subtract x from both sides to get: $-4x + 8 = 20$.

Step 2: Divide both sides by -4 to get $x - 2 = -5$.

Step 3: Add 2 from both sides to get $x = -3$.

Now let's reverse the process. Start with $x = -3$.

Step 1: Subtract 2 from both sides to get $x - 2 = -5$.

Step 2: Multiply both sides by -4 to get $-4x + 8 = -20$.

Step 3: Add x to both sides to get $-3x + 8 = -20 + x$.

Summary: To solve any linear equation, use these rules, not necessarily in the order listed. Practice develops a sense of the sequence that best leads to the solution:

1. Use the distributive law to remove parentheses.
2. Combine like terms so that each side of the equation is of the form $ax + b$.
3. Add the same expression to both sides of the equation so that x appears on only one side of the equation.
4. Divide by the nonzero coefficient of x; resulting in an equation of the form $x = c$.

Example 11.

Students will need to develop a facility for discovering mistakes in the procedure, when, after checking, it is discovered that the arrived at answer does not solve the original equation. In the following determine whether or not the following arguments are correct, and if incorrect, explain the error,

a.
$$\begin{aligned} 2(x+5) &= 13 \\ 2x+5 &= 13 \\ 2x &= 8 \\ 2x &= 8 \end{aligned}$$

b.
$$\begin{aligned} 3x-15 &= 24 \\ 3(x-5) &= 24 \\ x-5 &= \frac{24}{3} \\ x-5 &= 8 \\ x &= 13 \end{aligned}$$

c.
$$\begin{aligned} 2x+3 &= x+10 \\ 2\left(x+\frac{3}{2}\right) &= x+10 \\ x+\frac{3}{2} &= \frac{x}{2}+10 \\ \frac{x}{2} &= 10-\frac{3}{2} \\ x &= 10-3 = 7 \end{aligned}$$

Solution.

a. There is a mistake in the first step: the 2 is improperly distributed; the second line should be $2x + 10 = 13$. Now subtracting 10 from both sides gives $2x = 3$ which leads to the answer $x = 3/2$. Now check:

$$2\left(\frac{3}{2}+5\right) = 2\times\left(\frac{3}{2}\right) + 2\times 5 = 3 + 10 = 13$$

b. This is not so straightforward as the preceding problem, so first we check whether or not the answer satisfies the first equation: substitute 13 for x to get $3(13) - 15$ which is $39 - 15$, which simplifies to 24. Since this is the same as the right hand side of the equation, the answer is correct. But that does not mean that the argument is correct: we have to still check that the step from one line to the next is a correct application of arithmetic. In this case, it checks out: line one to two is the distributive property; line two to three: both sides are divided by 3; line three to four, replaces $24/3$ by the equivalent number 8 and finally we get to the last line by adding 5 to both sides.

c. First, let's check that 7 solves the first equation: with this substitution for x, the left hand side is $2(7) + 3 = 17$, and the right hand side is $7 + 10 = 17$. Now, let's check that the reasoning is correct. Well, there is a mistake in going from line two to three: we are dividing both sides by 2, but on the right, the second term (10) was not divided by two. The next step is correct: we have subtracted $x/2$ and $3/2$ from both sides. However, the step to the last line is faulty: in multiplying both sides by 2, we failed to multiply the term 10 by 2. This error has the effect of correcting the preceding error, so when we end up with $x = 7$, we accidentally ended up with the correct answer even though the steps were flawed.

Section 1.3: Creating and Solving Linear Equations to Model Real World Problems

Solve real world problems with one variable linear equations. 8.EE.7c

Up to this point in this chapter we have been discussing the questions: "What does an equation tell us?", "What does it mean to solve an equation?", and "What is the process for solving linear equations?" However our students are not likely to see an equation to solve except in other math or science classes that they will take or, eventually, will teach. So why is this material in the school curriculum?

The simplest answer is that humans are confronted with number-based issues many times in every day of their lives, and having a strong structural foundation to problem-solving provides confidence in the ability to deal with such problems. They will need tools to think through those problems in a way that leads to an acceptable course of action. If the course of action depends upon the values of specific quantities, it is highly likely that algebra - or more often, algebraic thinking - will be the tool to apply. An equally important reason is that many professions to which our students aspire require a methodology that is based upon mathematics. This is true in science, engineering, mathematics, medicine/pharmacy, accounting, finance, architecture. A strong mathematical foundation developed in middle school is essential to entering these fields, let alone succeeding in them.

It is coherent and effective algebraic thinking that is the goal of the study that started in sixth grade and continues through to high school graduation. It is *not* the protocol, or technique: we have computers and calculators to do that work. So then, the question again arises: why do we teach the protocol for solving equations? One might just as well ask: "Why does the plumber have to know his wrenches?", "Why does the dentist have to understand the dentist chair?", "Why does a carpenter have to know what a hammer can and cannot do?", "Why do we have to learn about our car before we can drive it?" The answers are so obvious that these questions hardly ever come up. But the connection between algebraic thinking and solving of real-life problems is not easily taken for granted, and one must be able to explain it - the teacher in order to scaffold the teaching, and the student in order to be motivated to learn.

Although this is the third section on linear equations, it is the most important one: the content here is the relationship between a problem stated verbally, and its algebraic restatement. This we call *mathematical modeling* of a problem. The state of one's financial resources may suggest that, while shopping, one take into consideration the relationship between cost and value. This may involve gathering of data (more shopping) and then analysis of those data. All of these operations involve algebraic thinking.

Modeling of a problem makes it possible to do calculations and make predictions. For example, we record the passage of time with clocks and watches, and make calculations based on those observations, conceptually using the model of a needle moving continuously along the real line in the positive direction. But we should not take the model too seriously: if we are told that the new record for running a mile is 3 minutes and 37.65 seconds, we should keep in mind that that is an approximation, as will be any response, no matter how accurate the timepiece. If in the next race, the same runner runs the mile in 3 minutes and 37.65 seconds according to the chronometer available, we say that the time is "the same," but we can neither claim that this is precise, nor that one race was run faster than the other - unless some other observer had a more precise chronometer.

Students should understand the power they have when they are able to move fluently between the verbal description and symbolic representation of a linear context. What does the symbolic representation allow them to do? Symbolic representations allow students to show the relationship between the quantities, solve problems, draw conclusions, make decisions, etc.. Of what must students be careful when they translate to a symbolic representation? Are they clear in what they have defined their variable to represent in the context? What quantities and units are involved? Once solved, can they interpret the solution in the context? Often, when students formulate the algebraic representation and solve it, they struggle to interpret that answer in its context and for confidence that they have even answered the question(s) being asked.

Being able to figure out how to attack problems as they come up, and what protocols to use to solve them, is the central goal of education. Students need to know how the protocols work in order to be able to formulate the problem so that a protocol can be applied. In the following set of problems we concentrate on how to get from a verbal problem to the equation that represents it.

Example 12.

Here is a game that illustrates the power of the algebra developed so far. I'm thinking of a number, and I'd like you to help me get it. Pick any number from 1 to 20. Add three and double the result. Add 8. Take half of that number. Subtract your original number. You've got the number I'm thinking of; it is 7, right?

Solution. Let us analyze this "trick." The subject picks a number; since it is a number but we do not know it, we call it N. Then the subject is asked to perform a set of operations, leading to the following list:

Operation	*Result*
Add 3 :	$N + 3$
Double :	$2(N + 3)$
Add 8 :	$2(N + 3) + 8$
Take half :	$N + 3 + 4$
Subtract original number :	$3 + 4$

So when you, the "magician," reveal that the number in hand is 7, we see that it is algebra, not magic. If this trick is played several times, it is a good idea to change the number 8 to another even number $2K$. Then the end result is "revealed" to be $3 + K$.

A variant of this that may seem a little more magical is this: at the $N + 3 + 4$ stage, ask for the number the subject now has. When this is received, mentally subtract 7 and says: "Your original number was . . . ."

Example 13.

a. Fred, Jon and Brody pooled their resources for a \$150 room at a resort. Jon put in twice as much as Fred, and Brody put in \$10 less than Fred. How much did each put in?

In this problem we are asked for the amount each put in the pool. In order to pick the "unknown," it may be necessary to read the problem more carefully. The statements all relate the unknowns to Fred's contribution, so we should select that as the primary unknown; let's call it F. Now James put in twice as much as Fred, so he put in twice F; that is, his contribution is $2F$. Brody put in \$10 less than Fred, so his contribution is $F - 10$. The sum of these numbers is \$150, and this becomes the equation:

$$F + 2F + (F - 10) = 150 .$$

We now assign this to our assistant, an expert on the preceding two sections of this chapter, who comes up with the answer : $F = 40$, so Fred put in \$40, Jon put in \$80, and Brody put in \$30.

Here is a slightly more complicated variant:

b. Fred, Jon and Brody pooled their resources for a \$150 room at a resort. Jon put in twice as much as Fred, and Brody put in \$10 less than Jon. How much did each put in?

We are still looking for three numbers, but it is no longer true that the other two are directly related to Fred's contribution. But Brody's is related to Jon's, and Jon's is related to Fred, so it still all comes down to Fred. Again, if we denote Fred's contribution as F, then that of Jon is $2F$. Now Brody put in \$10 less than Jon's, which is $2F$, so Brody's contribution is $2F - 10$. Now add them up to get \$150: $F + 2F + 2F - 10 = 150$, leading to the result $F = 32$.

This problem illustrates several possible confusions, the source of which is the ambiguity in various linquistic constructions. For example: statements like "A is four less than B" and "A is twice B." There is a significant difference between the statements "A is four less than B" and "A less four is B;" as there is between "A is twice B" and "twice A is B."

This problem with language can be resolved by testing with real numbers: take $A = 5$ and $B = 1$. Now, which is true: "5 less 4 is equal to 1" or is "5 is 4 less than 1"? The equation expressing the first statement is "5-4=1," which is true, and for the second we get "5=1-4," which is false. In the same way, we test the second statement: A is twice B. Take $A = 4$ and $B = 8$, put them in the statement, and pick the version that is true.

Example 14.

A salesman at the XYZ car dealership receives a salary of \$1,000 per month and an additional \$250 for each car sold. How many cars should he sell each month so as to earn \$8,000 in a month?

Solution. The way to the solution is to find out the relationship between income and number of cars sold. Test some numbers: If no cars are sold, the salary is just the base \$1000; if one care is sold, the earnings are \$1250. Look at some more test cases to discover the pattern:

a. if this salesman sells 4 cars, his income for that month is: 1000 + 4(250);

b. if this salesman sells 12 cars, his income for that month is: 1000 + 12(250);

c. if this salesman sells N cars, his income for that month is: 1000 + N(250).

The last expression simply amounts to recognizing that the computation for 4 or 12 or "no matter how many" cars sold is the same.

In this problem we want the income to be \$8,000, so we look at **c.**, set it equal to 8000 and hand it over to our algebra assistant for the solution.

Example 15.

Lucinda is on the school track team; she can run 8 miles in an hour. Her younger sister Josefa isn't yet a runner, but can walk at a pace of 3 miles an hour. Josefa left their home forty minutes ago heading toward downtown, and Lucinda wants to catch up with her. If she runs how long will it take to catch Josefa? How far will they be from home?

Solution. First, we focus on the first unknown that we have to determine: in this problem it is the time Lucinda needs to catch Josefa; let's call that T, in hours. In that time Josefa has walked $3T$ miles and Lucinda will have run $8T$ miles. So, after T hours, Lucinda is $8T$ miles from home; but since Josefa was already 2 miles down the road when Lucinda started, Josefa is $3T + 2$ miles from home. When they actually meet these two distances have to be the same, giving us the equation

$$8T = 3T + 2$$

The solution of this equation is $T = (2/5)$.

Now, it is important to recall what this means: what was T and what are the units for $2/5$? As soon as the algebra takes over, the meaning of symbols becomes irrelevant, but when we've solved the algebraic equation we must return to the meaning of the symbol to fully understand what we have discovered: T is $2/5$ *of an hour*, or 24 minutes. It takes energy, once the "math" is done, to return to focus to the problem. But when we do, we see there is another part of the problem: How far away have they gone. Well, Lucinda ran for 2/5 of an hour and she runs at 8 miles per hour, so she ran (2/5)8 miles, which is five and a third miles.

There is yet another subtlety in this solution, and it is important to recognize it and focus on it. In getting to the equation $8T = 3T + 2$. Where exactly does the 2 come from? T is the time Lucinda runs. The equation relates the distance (from the start) that Lucinda has run and that Josefa has walked. But Josefa started 40 minutes **before** $T = 0$. That is 2/3 of an hour, and 2/3 of the distance Josefa can walk in an hour (3 miles) is 2 miles. Therefore, at time T, Lucinda has run $8T$ miles, and Josefa has walked $3T$ miles *plus* the distance already covered: 2 miles.

Example 16.

My new hybrid car can get 35 miles to the gallon. Gas costs $3.25 per gallon. San Francisco is 825 miles from here. How much will I spend on gas to drive to San Francisco?

Solution. Let C be the cost of driving there. We have to relate C to miles, denoted by M, and the information we are given is how C relates to gallons, denoted by G, of gas: $C = 3.25G$ and how miles relate to gas: $M = 35G$. This tells us that $G = M/35$, and so we can substitute that in the first equation to get $C = 3.25(M/35)$, or $C = 0.093M$. Since we want to go 825 miles, the cost of gasoline will be $C = 3.25(825/35) = 3.25(23.57) = 76.761$.

The preceding example is, in part, an example of a *literal* problem: one in which several quantities are related and we want to express that relationship by a formula. So, in that example, we want to express the cost of gasoline (C) in terms of miles (M), already knowing the cost of gasoline per gallon and the number of miles per gallon; and we ended up with the relation $C = 0.093M$, or 9.3 cents per mile.

Here is another literal problem:

Example 17.

Let's return to Example 14, of the salesman at the XYZ dealership. We saw (see Example 14, part c)) that if the salesman sells N cars in a month, then his compensation is $1000 + 250N$. The salesman may ask: how many cars do I have to sell to have an income of C in a given month?

Solution. The relationship between compensation (C) and number of cars sold (N) is $C = 1000 + 250N$. The salesman wants to know what N should be to attain a certain value C, so he wants a formula that calculates N, given a value of C. These are the steps:

$C = 1000 + 250N$: this is the starting relationship.

$C - 1000 = 250N$, subtract 1000 from both sides.

$\frac{C - 1000}{250} = N$, divide both sides by 250.

This is the formula to calculate the number of cars to be sold to earn C dollars. So, for example, if the salesman wants to earn $25,000 in a particular month, he must sell $N = (25000 - 10000)/250 = 96$ cars.

To summarize: here is a procedure for solving problems:

1. Read the problem carefully, making sure to identify the unknown(s).
2. Recognize the information in the problem that can be translated into mathematical expressions or equations.
3. Apply the rules for solving linear equations.
4. Return to units to interpret your result.

EXAMPLE 18.

The conditions of John's job are that he can work whenever he wants to, but in any day he works, he receives no compensation for the first three hours of work, and $20 per hour for each subsequent hour. On one particular day he wants to buy a special shirt for $190. He has $70 in his pocket. How many hours must he work on that day to be able to buy the shirt?

SOLUTION.

Since John already has $70, he has to earn an additional $120 to buy a $190 shirt. At earnings of $20 per hour he therefore has to work 6 *paying* hours. But he gets no pay for the first 3, so he'll have to work 9 hours that day.

It looks like we used no algebra to solve this problem, but in actuality there is a lot of hidden algebra in the answer. Let's think more generally, as if we didn't know the price of the shirt to begin with. So, suppose in any day, John works N hours. Since he doesn't get paid for his first 3 hours, he puts in $N - 3$ paid hours, at $20/hour. Thus, he earns $\$20(N - 3)$ on a day when he works N hours.

In our particular problem, he starts with $70, and wants $190, so the number N of hours he has to work is the solution of the problem: $70 + 20(N - 3) = 190$.

Finally, the statement "he earns $\$20(N - 3)$ on a day when he works N hours." only makes sense if $N \geq 3$: he doesn't owe money on days when he works less than 3 hours. So, that statement about his earnings on any day has to be modified by adding "on a day when works at least 3 hours. Otherwise he earns nothing." This seems like an obvious point at this stage, but as we move toward the idea of function, we will have to remember that the formula has to be used only when it makes sense. To put this example in function notation, we should say that John's earnings are $20(N - 3)$ for $N \geq 3$ (the *domain* of the expression is $N \geq 3$).