

Chapter 1

Probability, Percent, Rational Number Equivalence

Traditionally, seventh grade starts by gathering up everything students have learned about numbers and arithmetic, in a way that increases their flexibility with operations while illuminating the underlying algebraic structure of the number system. Experience shows that this traditional approach is problematic: students, weak as well as strong, find this a dull review of either what they already know or what they are unlikely to learn through repetition. For this reason, cognizant that review is essential, it is desirable to embed this review in the introduction of a new topic. We choose probability as the new topic, with the idea that its extrinsic interest will attract the students' attention, while exhibiting the importance of arithmetic operations in context. Another reason for starting the year with basic probability activities is to develop a culture of thinking about mathematics as a way to investigate real world situations. A third reason is that such activities at the beginning of the year can help foster a classroom culture of discussion and collaboration.

Throughout this chapter students are provided with opportunities to review and build, based on knowledge from previous grades, fluency with fractions, percents, and decimals and recognize equivalent forms of rational numbers. Students should understand that fractions, percents and decimals are all relative to an agreement on what is the *whole* or *unit*. Students will also compare and order fractions (both positive and negative). The chapter concludes with a section specifically about solving percent and fraction problems, including those involving discounts, interest, taxes, tips, and percent increase or decrease. As students model mathematics, they begin to apply properties of operations (the "field axioms") informally, leading to the formalization in chapter 3.

This is students' first formal introduction to probability. They will have worked with data sets before, looking at various ways of representing the data to get the best information possible. In seventh grade, students have their first exposure to *statistics*, which can be described as the set of tools for the analysis of data. Probability lays the theoretical foundation for statistics; yet another reason to start 7th grade with this subject. We use statistics to come to a qualitative statement of what the data tell us about a certain population. So, a medical study of the effectiveness of medicine *A* in the treatment of disease *B* will end up with a conclusion of this kind: "the data suggest that *A* is 80% effective in curing subjects of disease *B*, while causing adverse side effects in 4% of the afflicted." Those percentages are found using the basic principles of probability, which consist of assigning numerical values to the probability of a specific set of outcomes of a situation (in this case, having contracted disease *B*).

In brief, probability is a branch of mathematics that provides the foundation for a statistical analysis of data. That analysis, however, is at the heart of what we call the *scientific method*: pose a hypothesis, gather data, analyze the data, estimate the likelihood that the hypothesis is verified.

This is students' first formal introduction to probability. In the first section students will study chance processes, experiments or situations for which they know the possible outcomes but do not know which outcome will occur at any run of the experiment. Students will look at probabilities as ratios, represented by fractions, decimals, or percents (part:whole). In Chapter 4 we will talk about part:part relationships where distinguishing between these

relations is going to be important. Eventually, it becomes clear that we can (depending upon where we want to have the emphasis) take either a part:whole or a part:part relationship and convert it to the other. For example, if $\frac{3}{5}$ of the class are girls we know that $\frac{2}{5}$ are boys and the ratio of girls to boys is 3:2. Later, in chapter 7, students will discuss “odds” which are part:part relationships.

Probabilities are often determined from the results or outcomes of experiments. They will learn that the set of all possible outcomes for an experiment is a *sample space*. They will recognize that the probability of any single *event* (a subset of the sample space) can be expressed in terms of impossible, unlikely, equally likely, likely, or certain or as a number between 0 and 1, inclusive. Students will focus on two concepts in the probability of an event: *experimental (empirical)* and *theoretical*. They will understand the commonalities and differences between experimental and theoretical probability in given situations. This will conclude the first section.

While studying probability, students continue their study of rational numbers. They will convert rational numbers to decimals and percents and will look at their placement on the number line. This lays the foundation for 8th grade where the *real line* representation of numbers leads to the realization of the existence of irrational numbers in order to complete the line. With this in mind, in the next section of this chapter, students solidify and practice rational number sense through the careful *review* of fractions, decimals and percents. The two key objectives of the second section are a) students should confidently articulate relationships among equivalent fractions, decimals, and percents using words, models, and symbols and b) students should understand and use models to find portions of different wholes.

The concept of equivalent fractions naturally leads students to the issues of ordering and estimation. Ordering positive and negative fractions will be connected to the number line. It is important that students develop estimation skills in conjunction with both ordering and operating on positive and negative rational numbers. Lastly, students look at percent as *per hundred*: a fraction with a denominator of 100. Percent and fraction contexts in this section will be approached intuitively with models.

The chapter concludes with a section in which students continue to solve contextual problems with fractions, decimals and percent but begin to transition from relying solely on models to writing numeric expressions. In subsequent chapters students will extend their understanding by writing equations and proportional equations using variables.

Section 1.1: Investigate Chance Processes. Develop/Use Probability Models

The mathematics emphasized in this chapter reflects the importance of probabilistic concepts in the world of today. References to probability are all around us, including weather forecasting. Suppose you have some outdoor plans made for a particular day and the weather report says that the chance of rain is 70%. Should you still go ahead with your plans or should you cancel them for another day? Another example of probabilities in daily life comes from the world of sports. A batting average involves calculating the probability of a player hitting the ball. That is, a batting average is a statistic ($\text{hits} \div \text{atbats}$) that is developed from past history. However, it is used in a theoretical sense: a batter with a .300 average is 50% more likely to get on base as a batter with a .200 average. So let's say your favorite baseball player is batting .300. This means that when he or she goes up to the plate, there is only a 30% chance of getting a hit! Playing the lottery is another instance of probability in real life. Millions of people around the world spend their money on lottery tickets in hopes of winning the big jackpot and becoming millionaires. But do these people realize how low their chances of winning actually are?

Probability is a vehicle for students to engage in a new mathematical topic while reviewing and practicing whole number and rational number arithmetic. We are also preparing the way for the study of statistical inference (Chapter 7), given that probability provides a mathematical description of randomness, such as the chance variation observed in the outcomes of randomized experiments and random samples. This development occurs as students consider and discuss with their peers the outcomes of a variety of probabilistic situations.

The origins of probability. In the 15th century, as mathematics was beginning to be realized as a vehicle for understanding how things work, attention was focused on finding ways to calculate the likelihood of events. Most historians think that this study originated in an unfinished dice game. The French mathematician Blaise Pascal received a letter from his friend Chevalier de Méré, a professional gambler, who attempted to make money gambling with dice. Chevalier de Méré's predicament involved two games of dice. In the first one, he made money by betting that he could roll a 6 on at least one of four consecutive rolls of a die. Empirical experience led him to believe that he would win more times than he would lose. He reasoned correctly that the chance of getting a six in one roll of a die is $\frac{1}{6}$. He then incorrectly thought that in four rolls of a die, the chance of getting one six would be $4 \times \frac{1}{6} = \frac{2}{3}$. Though his reasoning was faulty, he made considerable money over the years in making this bet. Today we know that the probability of winning this bet is $1 - \left(\frac{5}{6}\right)^4$, or 51.8%.

When folks would no longer bet on this proposition, de Méré modified the game by betting even money (original bet is either doubled or lost) that double 6's would turn up at least once in 24 throws of a pair of fair dice. This seemed like a good bet, but he began losing money. He reasoned correctly that the chance of getting a double six in rolling a pair of dice is $\frac{1}{36}$. However, he erred in thinking that in 24 rolls of a pair of dice, the chance of getting one double six would be $\frac{24}{36} = \frac{2}{3}$.

Based on empirical data (he lost a lot of money), he knew something was not quite right in the second game of dice. So he challenged his renowned friend Blaise Pascal to help him find an explanation. Pascal shared the problem with Pierre Fermat and together they solved the problem, which is often marked as the beginning of the era of the mathematical theory of probability.

Extension. One more good reason for starting with probability in a review of fraction arithmetic is that it provides a context for understanding the operations of addition and multiplication of fractions. The Chevalier's misunderstanding of addition led to his undoing. In the first instance he reasoned that, since the probability of rolling a six in one roll is $1/6$, then the probability of rolling a six in two rolls is twice that: $1/3$, and in four rolls, $2/3$. Apparently, he assumed that in repeating an experiment, the probability of at least one favorable outcome increases, so the probabilities must add. If he extended that one more time he would have seen that that assumption is in error, for the probability of rolling a six in 3 rolls would be $1/3+1/3+1/3 = 1$. But we know that it is quite possible to roll a die 3 times and not get a six; in fact it is conceivable that we could roll the die 500 times and never see a six.

When he went to Pascal to find out what was wrong (and how to make it right). Pascal surely did this mind experiment and saw fairly soon that, in a question like this, probabilities do not add. But then, what? His thinking might have gone this way: it is pretty clear that, when we repeat an experiment, the probabilities of a favorable outcome every time multiply. For example, suppose we ask: what is the probability of getting a six in two consecutive rolls? Well, the probability of rolling a six in one roll is $1/6$. So one-sixth of the time (on average) we roll a six on the first roll. Now, for the second roll, only those one-sixth of the rolls count: we will roll a second six one-sixth of the times we have already rolled a six. $1/6$ of $1/6$ is $1/36$.

Pascal moved forward with this reasoning: if we want a certain outcome to happen every time in a series of experiments, then the probabilities multiply. But, we are not asking for success *every* time, we are asking only for one success in the series. Surely the probability increases as we increase the number of rolls, but probabilities do not add. Then how *do* we calculate the probability? Here comes the genius of Pascal: he asked: what is the probability that the Chevalier loses? Well, it is the probability of rolling a 1,2,3,4 or 5 in every one of the four consecutive rolls. The probability of rolling any number other than a six is $5/6$, therefore, the probability of *not* rolling a six in four consecutive rolls is $(5/6)^4$. The Chevalier wins precisely when he does not lose; so the probability of rolling at least one six in four consecutive rolls is $1 - (5/6)^4 = 0.517747$. So, even though his thinking was faulty, his betting strategy was winning: in 1000 games, he could expect to win about 518 of them, so he could keep playing the game.

In the second game, in a roll of a pair of dice, there are a total of 36 possible outcomes (the six outcomes of the first die combined with each of the six outcomes of the second die). Out of these 36 outcomes, only one of them is a double six. So, the probability of getting a double six is $\frac{1}{36}$ in rolling a pair of dice. Likewise, the probability of not getting a double six is $\frac{35}{36}$. The probability of getting no double six in 24 rolls of a pair of dice is:

$$\begin{aligned}
P(\text{at least one double six in 24 rolls}) &= 1 - P(\text{no double six in 24 rolls}) \\
&= 1 - \left(\frac{35}{36}\right)^{24} \\
&= 0.4914
\end{aligned}$$

Thus the probability of getting at least one double six in 24 rolls of a pair of fair dice is 0.4914. On average, de Méré would only win about 49 games out of 100 and his opponent would win about 51 games out of 100 games. Clearly, the Chevalier would, in the long run, lose and lose big. And he did.

As we have seen above, when we repeat an experiment, looking for a certain outcome *every* time, probabilities multiply, but if we look for that outcome at least once in the series, they do not add. Then, when do probabilities add? Let's examine some situations in the one die-rolling experiment. What is the probability of rolling an odd number? In the sequence {1, 2, 3, 4, 5, 6} there are three odd numbers and three even numbers, so we should expect that half the time we roll an odd number: that is, the probability is 1/2. Notice that this is (1/6) + (1/6) + (1/6); that is the sum of the probabilities of rolling a 1 or a 3 or a 5. What is the probability of rolling a prime number? Again, there are three primes: {2, 3, 5}, so the probability is 1/2, which is the sum of the probabilities of rolling a 2, 3 or 5. So, here again probabilities are adding. Now let's ask: what is the probability of rolling an odd number *or* a prime. That means, rolling anything but a 4 or a 6, so the probability is 2/3. Since 2/3 ≠ 1/2 + 1/2, probabilities now don't add. The reason for this is that the two different questions have the answers 3,5 in common, so those answers have been counted twice. Compensating for that, we have 2/3 = (1/2 + 1/2) - 1/3: a true statement!

This is a long story which we cannot complete here; but we can here begin the discussion. The important thing for students is that, in calculating probabilities, addition is much more difficult than multiplication - just as it is for fractions.

End Extension.

Basic concepts. Probability is about the likelihood of the occurrence of event; attempting to put a numerical measure of the chance that it will occur. In probability, we study chance processes, which concern experiments or situations where we know which outcomes are possible. An *experiment* is an activity whose results can be observed and recorded. Each of the possible results of an experiment is an *outcome*. If we toss a fair coin (*i.e. heads and tails are equally likely to occur therefore it's a fair coin*), there are two distinct possible outcomes: head (*H*) and tails (*T*).

The set of all possible outcomes for an experiment is a *sample space*. The sample space *S* for rolling a fair die is $S = \{1, 2, 3, 4, 5, 6\}$. An *event* is a collection of outcomes, a set in the sample space. The set of all even-numbered rolls {2, 4, 6} is a subset of all possible rolls of a die {1, 2, 3, 4, 5, 6} and is an event.

So, how do you measure the chance of an event? There are two ways: theoretical and experimental. By *theoretical*, we refer to what can be deduced through logical thinking. By *experimental* we refer to what is observed as we repeat the experiment over and over. Let's develop these concepts, starting with this question: What is the probability of rolling a six with a fair die?

With a roll of a fair die, there are six possible outcomes 1, 2, 3, 4, 5, 6. Each outcome is one out of six equally likely outcomes; we express this by writing that each outcome has a probability of $\frac{1}{6}$. Hence, the *theoretical probability* of rolling a six with a fair die is $\frac{1}{6}$. This suggests that the correct definition of the probability of an event (where all outcomes are equally likely) is:

$$p(E) = \frac{\text{Number of Outcomes in the Event}}{\text{Number of Possible Outcomes}} .$$

We may now actually roll the die a number of times, and keep track of the number of times the desired event (a 6)

occurs. This leads to what we describe as the *experimental probability*, the fraction of favorable occurrences over all occurrences :

$$\frac{\text{Number of Observed Occurrences of the Event}}{\text{Total Number of Trials}}$$

Although it would be impossible to conduct an infinite number of trials, we can consider the long-run relative frequency as an increasingly close approximation to the actual probability or the theoretical probability as the size of the data set (sample) increases. This is referred to as the *Law of large numbers*. It is also known as Bernoulli's theorem, in honor of Jakob Bernoulli (1654-1705). How many trials are enough? It depends upon how sure you want to be that the die is fair: if you want 80% confidence, you will have to roll the die a lot more than if you are satisfied with 60% confidence.

Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability. For example, when rolling a number cube 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times. 7.SP.6

An investigation regarding the law of large numbers was conducted by John Kerrich during World War II. In April of 1940, while visiting family in Copenhagen, John Kerrich was caught in the Nazi invasion and was imprisoned. To pass time, Kerrich tossed a coin 10,000 times. On his release Kerrich published an account of his experiments in a short book entitled *An Experimental Introduction to the Theory of Probability*. A sample of his results is in Table 1.1. The relative frequency column on the right is obtained by dividing the number of heads by the number of tosses of the coin.

Number of tosses	Number of Heads	Relative Frequency
10	4	0.400
50	25	0.500
100	44	0.440
500	255	0.510
1,000	502	0.502
5,000	4,034	0.504
10,000	5,067	0.507

Table 1

As the number of tosses increased, Kerrich obtained heads close to half the time. The long-run relative frequency for Kerrich's tosses gives a result of $5,067/10,000$, or approximately $\frac{1}{2}$.

Some probabilities cannot be determined by the analysis of possible outcomes of an event; instead, they can only be determined through gathering empirical data. Why? Because the outcomes may not be equally likely and we have no way of determining the likelihood of the outcomes without experimentation. Another reason could be that there are many more factors involved over which we have no control (as in weather forecasting). As another example, consider an experiment to see how often a flipped Hershey's Kiss or a thumb-tack will land on its base.

In Chapter 7 we will see how to use statistics to estimate probabilities in a given population by examining a sample of the population. Generalizations about a population from a sample are valid only if the sample is representative of that population, and a sample of 1000 people provides more reliable and convincing data about the larger population than does a survey of 5 people. Subsequently the larger the number of trials (people surveyed), the more confident you can be that the data reflect the larger population.

Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around 1/2 indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event. 7.SP.5

Previously we made a statement that probability is about how likely an event is. Let us further explore this idea through some examples.

EXAMPLE 1.

A fair die is tossed. Let $S = 1, 2, 3, 4, 5, 6$. Let's calculate each of the following probabilities.

- a. the event A that the outcome is a 5;
- b. the event B that the outcome is an even number;
- c. the event C that the outcome is a number greater than 20;
- d. the event D that the outcome is a number less than 20.

SOLUTION. The expression $P(X)$ represents the probability (or likelihood or chance) that event X will occur. You can quantify probability with a fraction, percent, or decimal number. Given, each of the 6 numbers in set S has an equal chance of being rolled,

a. If we replace X by A , the outcome is 5, then $P(A) = \frac{1}{6}$.

b. If we replace X by $B = \{2, 4, 6\}$, then $P(B) = \frac{3}{6} = \frac{1}{2}$.

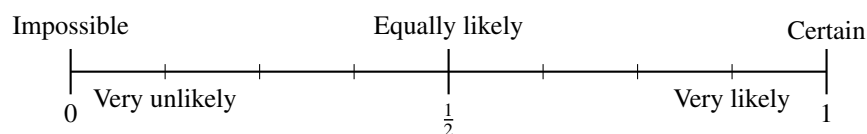
Why is $P(B) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$, but in de Méré's case this calculation failed? We will explore this further in the next few pages.

c. The event C is impossible, since the largest possible outcome is 6. Thus the set C has no members (this is called the *empty set*, denoted by \emptyset), then $P(C) = \frac{0}{6} = 0$.

d. If we replace X by D , the event D is certain to occur. D is the whole sample space, S , so we say the chance of D happening is 1, that is $D = S$, so $P(D) = \frac{6}{6} = 1$

To clarify, in Example 1(c), event C is the empty set. An event such as C that has no outcomes is an *impossible event* and has a probability of 0. In Example 1(d), event D consists of rolling a number less than 20. Because every number in S is less than 20, the $P(D) = 1$. An event that has a probability 1 is a *certain event*.

The number line is an important visual model of how likely an event can be.



In summation, the likelihood that an event will occur is expressed by a number called the *probability* of an event, where the probability ranges from 0 (impossibility) to 1 (certainty), or equivalently, when they are given as percentages, between 0% and 100%. A probability of 0% means that the event cannot possibly occur, as seen in Example 1(c). A probability of 100% means that the event is certain to occur, as seen in Example 1(d), and the greater the probability, the more likely an event is to occur. If we flip a coin, the probability that it will land up is 0.5 or 50%, since we expect, either on the basis of the coin's symmetry or by data gathered from past experience, that half the time we obtain heads and half the time tails.

A probability of 50% means the event is as likely to occur as not to occur. In particular, we note:

If A is any event and S is the sample space, then $0 \leq P(A) \leq 1$.

The discussion with regards to a fair die being tossed, resulted in distinct possible outcomes of a chance process that were all equally likely. If each possible outcome is equally likely, then we call a probability model for such a process a *uniform probability model*. If there are n possible outcomes, the probability of each outcome is $\frac{1}{n}$.

Consider again Example 1. A toss of the die results in any of the sides equally likely to land face up. So the probability of rolling a 1 is equal to the probability of rolling a 2, which is equal to the probability of rolling a 3, and so on. That is,

$$P(\text{roll a 1}) = P(\text{roll a 2}) = P(\text{roll a 3}) = P(\text{roll a 4}) = P(\text{roll a 5}) = P(\text{roll a 6}).$$

Mathematically this means that given the sample space $S = \{x_1, x_2, \dots, x_k\}$, and outcomes that are equally likely, then $P(x_1) = P(x_2) = \dots = P(x_k)$, where $P(x)$ represents the probability of outcome x .

Additionally, notice

$$P(\text{roll a 1}) = \frac{1}{6}, P(\text{roll a 2}) = \frac{1}{6}, P(\text{roll a 3}) = \frac{1}{6}, P(\text{roll a 4}) = \frac{1}{6}, P(\text{roll a 5}) = \frac{1}{6}, P(\text{roll a 6}) = \frac{1}{6}.$$

When you roll a fair die, the probability of rolling a 1 or a 2 or a 3 or a 4 or a 5 or a 6 (Example 1(d)) the result is 1. Therefore,

$$P(\text{roll a 1}) + P(\text{roll a 2}) + P(\text{roll a 3}) + P(\text{roll a 4}) + P(\text{roll a 5}) + P(\text{roll a 6}) = 1,$$

hence

$$P(x_1) + P(x_2) + \dots + P(x_k) = 1.$$

EXAMPLE 2.

Let's consider additional questions relating to Example 1. A fair die is tossed. Let $S = \{1, 2, 3, 4, 5, 6\}$. Calculate each of the following probabilities. .

- the event that the outcome is a 1, 2, 3, 4, or 6;
- the event that the outcome is a 5.

SOLUTION.

- Since the outcomes 1, 2, 3, 4, or 6 are distinct possible outcomes, then the probability of rolling a 1, 2, 3, 4, or 6 is the sum of the probabilities of rolling a 1, of rolling a 2, of rolling a 3, of rolling a 4, and of rolling a 6. That is, $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{5}{6}$, so the $P(1, 2, 3, 4, \text{ or } 6) = \frac{5}{6}$
- $P(5) = \frac{1}{6}$

Yet, is there another way of looking at the probability of rolling a 5, given the information gathered in finding the probability of rolling a 1, 2, 3, 4, or 6? In Example 2(b), let's rephrase the question.

Calculate the probability of the event that the outcome is not a 5.

Given that the $P(\text{roll a 1}) + P(\text{roll a 2}) + P(\text{roll a 3}) + P(\text{roll a 4}) + P(\text{roll a 5}) + P(\text{roll a 6}) = 1$, then $P(1, 2, 3, 4, \text{ or } 6) + P(5) = 1$, which results in the $P(5) = 1 - P(1, 2, 3, 4, \text{ or } 6)$. Are we at all surprised at the

outcome? No, because we've calculated $P(5)$ before. More importantly, we should have used this logic to calculate the probability that the outcome is a 1,2,3,4,6. This strategy, computing the probability of an event happening by instead computing the event it won't happen, and subtracting that from 1, is often the easiest way to proceed. Let us summarize the principles behind this.

Two events are *mutually exclusive* if they have no outcomes in common. For instance, in Example 1, event A (rolling a 5) and event B (rolling an even number: 2, 4, 6) are mutually exclusive events because no outcome is in common to both events. If we say that event F is rolling a number less than 5 (namely 1, 2, 3, 4), then events B and F would not be mutually exclusive because they have outcomes in common (rolling a 2 or 4).

Two events, X and Y , are *complementary* if they are mutually exclusive and together they make up the entire sample space.. Said another way, X and Y are complementary if everything in the sample space not in X is in Y . Because of this, the complement of X is referred to as “not X .” Therefore,

$$P(X) + P(\text{not } X) = 1 \quad \text{so} \quad P(\text{not } X) = 1 - P(X) \quad \text{and} \quad P(X) = 1 - P(\text{not } X) .$$

Nonetheless, the example shows that counting the complementary event is especially effective for counting where an “at least” condition must be satisfied, although in Chapter 1 students will be looking for a pattern or structure to emerge.

EXAMPLE 3.

Flip a coin four times. What is the probability of getting at least one head?

SOLUTION. A *tree diagram* and an *organized list* can be constructed as in Figure 1. For each of the two outcomes for the first flip, the tossing of the coin a second time has two possible outcomes (because the outcome of flipping the coin a second time does not depend on the outcome of flipping the coin the first time), and so on. So all together, there are $2 \times 2 \times 2 \times 2 = 2^4 = 16$ ways in which the tosses may occur. There are many ways in which one or more heads can occur. However, the complementary event is when no heads occur. There is just one way to get no head, namely, if all 4 tosses are tails. Thus, there are $16 - 1 = 15$ ways in which at least one head appears in the four tosses. Therefore, the probability of getting at least one head is $\frac{15}{16}$.

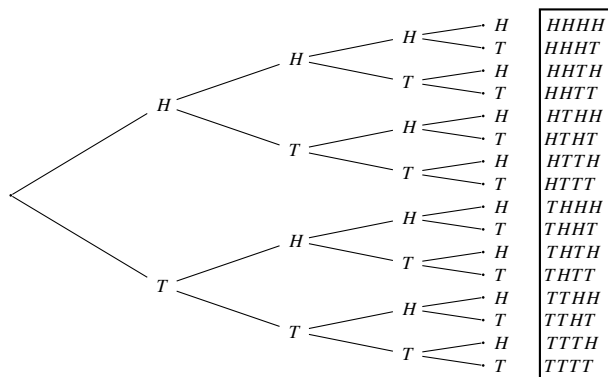


Figure 1

EXAMPLE 4.

Flip a coin four times. What is the probability of getting heads for all four flips? One approach is to think in terms of a sequence of flips, where as soon as you get tails, you stop.

SOLUTION.

1. The probability of coming up heads on the first flip is $1/2$. If you get tails on the first flip, you might as well stop, because you cannot possibly get four heads. So, half the time you stop, and half the time you keep going.
2. Assuming we kept going, then we flip the coin a second time. Again, the probability of heads is $1/2$. Again, we only keep going if it comes up heads. So half the time we keep going. Overall, the chance that we will keep going is $1/2$ of $1/2$, or $1/4$.
3. By now, $3/4$ of the time we will have stopped, and $1/4$ of the time we will have moved on to flip the coin a third time. Again, the probability of heads is $1/2$. So, the probability that we will keep going is $1/2$ of $1/4$, or $1/8$.
4. Finally, we have the fourth coin flip. We only get to this point $1/8$ of the time. Again, the probability of heads is $1/2$. The probability of four heads is $1/2$ of $1/8$, or $1/16$.

As a shortcut, we could say that the probability of getting heads on any one throw is $1/2$. The probability of getting four heads in a row therefore is $(1/2)(1/2)(1/2)(1/2)$, or $(1/2)^4$. This example engages students in conjecturing about the multiplication rule of probability, an intuitively plausible, but more advanced probability technique developed in high school. Additionally, it is a better method, for it is less tedious than drawing a tree diagram. In particular, using example 2, the probability of getting at least one “tails” in four flips is $15/16$ (because “tails” and “heads” are interchangeably probable). So, we can conclude directly from Example 2, that the probability of *not* getting one “tails,” which is the same as getting four heads is $1/16$.

In the above example, the four-event experiment (or *multistage experiment*) requires four actions to determine an outcome. This is an example of a compound event: when a particular experiment is executed two or more times. In this case there is a question to consider. Does the occurrence of the event in one stage have an effect on the occurrence of the event in the other? An important factor in calculating probabilities for multistage experiments is the distinction between *independent* and *dependent* outcomes at a stage. If you flipped a coin 10 times in a row and all 10 flips came up heads, would you think that your next flip is more likely to be a tail because a tail is “due”? Saying “a tail is due”, or “just one more go, my luck is due” is called The Gambler’s Fallacy. Some people think “it is overdue for a tail”, but a coin does not “know” it came up heads before; hence the next toss of the coin is totally independent of any previous tosses. In fact, on your next flip you are just as likely to get a tail as a head. There is nothing in the flipping history of the coin that can influence the next flip. The outcome of one coin flip is independent of the outcome of any other coin flip. In other words, the occurrence or nonoccurrence of one event has no effect on the other. In contrast, a dependent event is a second event whose result depends on the result of a first event, such as taking out a marble from a bag containing some marbles and not replacing it, and then taking out a second marble.

EXAMPLE 5.

This illustration of “dependence” is that of extracting marbles from a bag. Suppose that a bag contains 16 marbles, 9 of which are green and 7 are red. The experiment is that of extracting a marble from the bag. We can see (taking the quotient of favorable outcomes to possible outcomes) that the probability that the extracted marble is green is $9/16$, and that it is red is $7/16$. Now, suppose the experiment is repeated a second time. The probabilities now depend upon whether or not the marble extracted the first time is replaced. If it is, the probabilities of the color of the second marble are the same as the first. But if the marble is not replaced, the probability changes, depending upon the color of the first extracted marble. If it was green, then there are 15 marbles in the bag, 8 green and 7 red; and if it was red, there are 9 greens and 6 reds. This is an issue that will be explored further in high school; in 7th grade, students should see and understand the difference.

We can use the organized list and tree diagram of Figure 1 to calculate other probabilities as well. For example, what is the probability of getting *exactly* three heads and one tail when we toss the coin four times? Three heads and one tail occur in 4 of the 16 outcomes - namely, *HHHT*, *HHTH*, *HTHH*, and *THHH*. So the probability that either one of these four outcomes will occur is

$$\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{4}{16} = \frac{1}{4}$$

Determining the probability of the outcome of getting exactly three heads and one tail allows students to explore and conjecture about the addition rule of probability as they continue to refine their understanding.

Notice that the probability of getting *all* four heads versus the probability of getting *exactly* three heads and one tail versus the probability of getting at least one head represents an unfair game in which the events are not equally likely. All sixteen outcomes are equally likely, however the probability of getting *at least one* head versus getting *exactly* 3 heads and one tail reveals that for three heads, *HHHT* is different from *HHTH* because order matters. If we were to flip four coins at once, then *HHHT* is the same as *HHTH*. This distinction will be made clear in secondary mathematics.

In order for students to determine how likely an event is, it is important that all possible outcomes are generated – for example, by modeling, using tools such as organized lists, tables, histograms or tree diagrams. This builds a strong foundation for the more advanced probability techniques that will develop in 7th grade and beyond.

Section 1.2: Equivalence and Conversion in Rational Number Forms (fraction, decimal, percent).

In this section students solidify and practice rational number sense through the careful *review* of fractions, decimals and percent. The two key objectives of this section are a) students should be confidently able to articulate with words, models and symbols, relationship among equivalent fractions, decimals, and percent and b) students should understand and use models to find portions of different wholes.

The concept of equivalent fractions naturally leads students to the issues of ordering and estimation, and ordering positive and negative fractions will be connected to the number line. It is important that students develop estimation skills in conjunction with both ordering and operating on positive and negative rational numbers.

Lastly, students look at percent as being a fraction with a denominator of 100. Percent and fraction contexts in this section will be approached intuitively with models. In Section 1.3 students will begin to transition to writing numeric expressions.

In Grade 6 students learn that the number line expands to the left of zero, exploring integers and negative fractions and decimals. They learn how to place them on the number line and how to compare numbers. They get a sense, in terms of the number line, of the interpretation of the fraction p/q as the adjunction of p copies of the line segment of length $1/q$. In Grade 7 the students learn how to represent arithmetic operations on the number line, and understand that the rules of arithmetic as they know them, extend to the system of rational numbers.

Seventh grade, to a great extent, is retrospective; that is, we focus on the structure of the mathematics already learned and how the (pictorial/concrete) models reveal how quantities are related. On the other hand, Eighth grade is summative: by exploring operations with numbers as points on the number line, students are naturally led to irrational numbers and begin to get a sense of the completeness of the real-number system.

The word fraction comes from the Latin “fractio” or “fractus” meaning “to break” or “broken.” The use of fractions began with human observations of nature to express quantities that were less than a whole unit, such as divisions of the day. As early as 2000 BCE., the Babylonians used fractions; however, the denominators of their fractions were always powers of 60, corresponding to their base system. Evidence of the early stages in the development of fractions can be found in Egyptian mathematics in the Rhind Mathematical Papyrus: out of 87 problems on the Rhind, only 6 did not involve fractions.

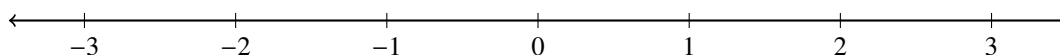
Fraction ideas appear to have been used in many cultures. Our method of writing fractions can be attributed to the

Hindus, most notably the Indian mathematician Aryabhata. By the year 1000 CE, Arabs had introduced the use of the fraction bar in their writings. Fibonacci was the first European mathematician to use the fraction bar as it is used today.

In early grades students study whole numbers as corresponding to points on the number line. Interpreting numbers as points on the number line allows for fractions to be seen as measurements with new units, created by partitioning the whole number unit into equal pieces. Recall that a fraction is a point on the number line represented by the quotient of a whole number by a counting number; a *rational number* is then a point on the number line represented by a quotient of an integer by a counting number. Therefore, integers and then the rational numbers are associated to points in the line.

It is instructive to recall the actual process of identifying number with a point on the line. With a straight edge, draw a horizontal line. Given any two points a and b on the line, we say that $a < b$ if a is to the left of b . The piece of the line between a and b is called the interval between a and b . It is important to notice that for two different points a and b we must have either $a < b$ or $b < a$. Also, recall that if $a < b$ we may also write that as $b > a$.

Pick a point on a horizontal line, mark it and call it the origin, denoted by 0. Now place a ruler with its left end at 0. Pick another point (this may be the 1 cm or 1in point on the ruler) to the right of 0 and denote it as 1. We also say that the length of the interval between 0 and 1 is one unit. Mark the same distance to the right of 1, and designate that endpoint as 2. Continuing on in this way we can associate to each positive number a point on the line. Now mark off a succession of equally spaced points on the line that lie to the left of 0 and denote them consecutively as $-1, -2, -3, \dots$. In this way we can imagine all integers placed on the line.



We can associate a half integer to the midpoint of any interval; so that the midpoint of the interval between 3 and 4 is 3.5, and the midpoint of the interval between -7 and -6 is -6.5 . If we divide the unit interval into three parts, then the first part is a length corresponding to $1/3$, the first and second parts correspond to $2/3$, and indeed, for any integer p , by putting p copies end to end on the real line (on the right of the origin if $p > 0$, and on the left if $p < 0$), we get to the interval representing $p/3$. We can replace 3 by any positive integer q , by constructing a length which is one q th of the unit interval. In this way we can identify every rational number p/q with a point on the horizontal line, to the left of the origin if p/q is negative, and to the right if positive.

Note that, for any q , if p is a multiple of q , say $p = nq$, (with n an integer), then the point corresponding to the fraction p/q is precisely the integer point n . More generally, observe that if p/q and r/s are equivalent fractions, then the points corresponding to p/q and r/s are the same point.

Extension

In the discussion above, we describe how to find points on the number line to represent fractions p/q . This requires partitioning the unit interval on the number line into q equal parts. Students may be interested in the construction that accomplishes that. On the next page, Figure 2 illustrates how to do that with $q = 5$. In that figure, the line segment AB represents the unit interval on the real line, with A at the origin and B at 1. Now draw another line passing through the point A . On that new line mark off 5 copies of any length as shown; however if one picks a length that appears to be close to $1/5$, the construction will be easier. Now draw a line from the last point (denoted “5” in the figure) to the point B . Draw lines parallel to that line from each of the the division points 1, 2, 3, 4. Then those lines will intersect the interval AB in points that segment AB in 5 equal portions. Of course “5” can be replaced by any whole number q .

Actually doing this easily persuades that the construction works. The reason why it works is called *The Fundamental Theorem of Similarity* and will be introduced in grade 8 and fully explained in grade 10.

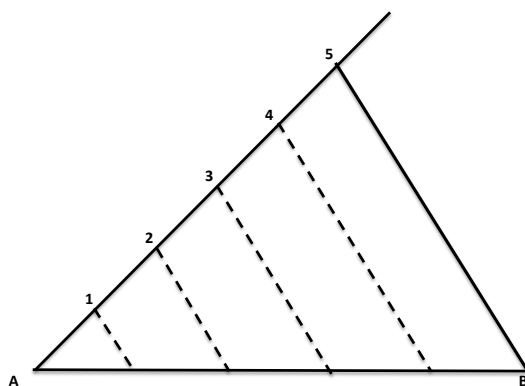
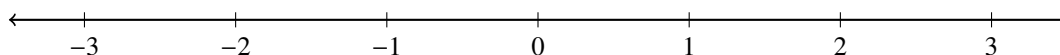


Figure 2

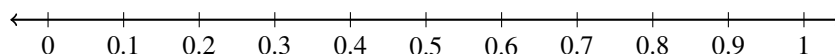
End Extension

Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats. 7.NS.2d

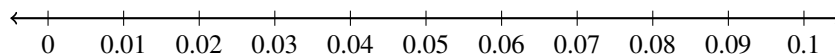
Decimals are special fractions, those with denominator 10, 100, 1000, etc. We think about decimals as “filling in” the locations on the number line between the whole numbers. We can think of plotting decimals on the number line in successive stages. At the first stage, the whole numbers are placed on a number line so that consecutive whole numbers are one unit apart.



At the second stage the decimals that have entries in the tenths place, are spaced equally between the whole numbers, breaking each interval between consecutive whole numbers into 10 smaller intervals each one-tenth unit long. Enlarging the interval between 0 and 1, we get:



And so on. Enlarging the interval between 0 and 0.1, we get:



We can think of the stages as continuing indefinitely. The digits in a decimal are like an address. When we read a decimal from left to right, we get more and more detailed information about where the decimal is located on the number line. These finer and finer partitions constitute a sort of address system for numbers on the number line: 0.543 is, first, in the neighborhood between 0.5 and 0.6, then in part of that neighborhood between 0.54 and 0.55, then exactly at 0.543. It’s very similar to specifying a geographic location by giving the country, state, county, zip code, street, and street number.

Now, what points on the line are represented by terminating decimals?

Rational numbers that have finite decimal expansions (terminating decimals) can be found using long division; we find that such a number can be expressed as a fraction whose denominator is a base 10 unit (10, 100, 1000, etc.):

$$\frac{a}{b} = \frac{n}{10} \quad \text{or} \quad \frac{n}{100} \quad \text{or} \quad \frac{n}{1000} \quad \text{or} \quad \dots$$

for some whole number n .

In this case then,

$$\frac{10a}{b} = n \quad \text{or} \quad \frac{100a}{b} = n \quad \text{or} \quad \frac{1000a}{b} = n \quad \text{and so on} \dots$$

We can find the whole number n by dividing b successively into $10a$, $100a$, $1000a$, and so on until there is no remainder and the process terminates (at some point there is no interval left over). The point is represented by a terminating decimal.

How do we find the decimal expansion of a rational number p/q ?

EXAMPLE 6.

Express $\frac{7}{16}$ as a decimal using long division.

SOLUTION.

$$\begin{array}{r} 4 \\ 16 \overline{)70} \\ \underline{64} \\ 6 \end{array}$$

$$\begin{array}{r} 43 \\ 16 \overline{)700} \\ \underline{640} \\ 60 \\ \underline{48} \\ 12 \end{array}$$

$$\begin{array}{r} 437 \\ 16 \overline{)7000} \\ \underline{6400} \\ 600 \\ \underline{480} \\ 120 \\ \underline{112} \\ 8 \end{array}$$

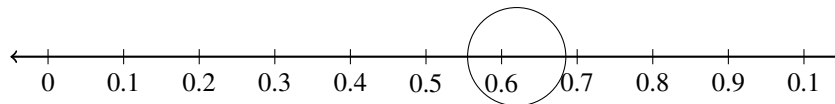
$$\begin{array}{r} 4375 \\ 16 \overline{)70000} \\ \underline{64000} \\ 6000 \\ \underline{4800} \\ 1200 \\ \underline{1120} \\ 80 \\ \underline{80} \\ 0 \end{array}$$

As illustrated; $70,000 = 16 \times 4375$, which means that $\frac{7}{16} = \frac{4375}{10000} = .04375$.

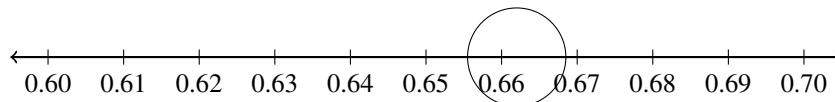
The finite decimals are the rational numbers that eventually come to fall exactly on one of the tick marks in this decimal address system. But now, this doesn't always work.

EXAMPLE 7.

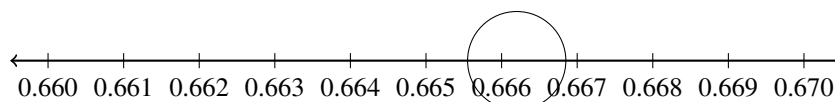
$\frac{2}{3}$ is always sitting two-thirds of the way along the third subdivision.



and then

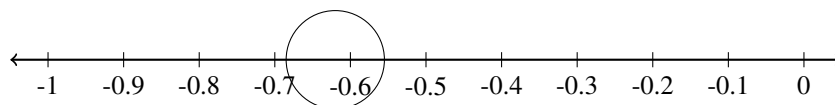


and so on...

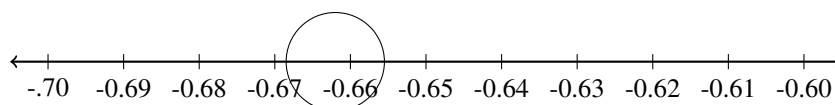


It is 0.66 plus two-thirds of a thousandth, and 0.666 plus two-thirds of ten-thousandth, and so on. The decimals 0.6, 0.66, 0.666 are successfully closer and closer approximations to $\frac{2}{3}$, that is, we say $\frac{2}{3}$ has an infinite decimal expansion consisting of entirely 6's in which we place a bar over the 6 to indicate that it repeats indefinitely: $0.\overline{6}$.

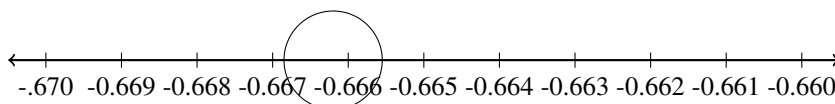
With the case of negative rational numbers the progression would be as follows; $-\frac{2}{3}$ is always sitting two-thirds of the way along the third subdivision. It is -0.66 minus two-third of a thousandth, and -0.666 minus two-third of ten-thousandth, and so on. The decimals -0.66 , -0.666 , -0.6666 are successfully closer and closer approximations to $-\frac{2}{3}$, that is, we say $-\frac{2}{3}$ has an infinite decimal expansion consisting of entirely 6's in which we place a bar over the 6 to indicate that it repeats indefinitely : $\frac{2}{3} = 0.\overline{6}$.



and then...



and finally...



How do we know if the fraction will terminate or repeat?

First let us look at terminating decimals. A terminating decimal, like .275, is a sum of fractions where each denominator is a power of ten (prime factors 2's and 5's). So,

$$.275 = \frac{2}{10} + \frac{7}{100} + \frac{5}{1000} = \frac{2}{10} + \frac{7}{10^2} + \frac{5}{10^3}.$$

Now, if we put these terms over a common denominator, we get

$$\frac{2(10)^2 + 7(10) + 5}{10^3} = \frac{275}{10^3}$$

In the same way, .67321 becomes

$$\frac{67321}{10^5},$$

In general a terminating decimal leads to a fraction of the form $A/10^e$ where A is an integer and e is a positive integer. So, if a decimal terminates after e terms, it can be written as a fraction with denominator 10^e . Now, since $10 = 2 \cdot 5$, we can write this as $2^e \cdot 5^e$, and we conclude that a terminating decimal can be written as a fraction whose denominator has prime factors only 2 and 5. On the other hand, we see that any fraction whose denominator (in simplest form) is a product of 2's and 5's has a terminating decimal. Let $A/(2^a 5^b)$ be such a fraction (A is an integer). Suppose $a = b$. Then our fraction is

$$\frac{A}{2^a \cdot 5^a} = \frac{A}{10^a}$$

so the decimal terminates at the a th place. If on the other hand $a \neq b$ we can multiply numerator or denominator by a power of 2 or 5 so as to get both exponents in the denominator the same. Let's illustrate:

$$\frac{A}{2^6 \cdot 5^9} = \frac{2^3}{2^3} \frac{A}{2^6 \cdot 5^9} = \frac{8A}{2^9 5^9} = \frac{8A}{10^9},$$

$$\frac{A}{2^6 \cdot 5^4} = \frac{5^2}{5^2} \frac{A}{2^6 \cdot 5^4} = \frac{25A}{2^6 5^6} = \frac{25A}{10^6}.$$

This discussion may be difficult for students to understand, as they have not yet been exposed to integer exponents (this comes up in eighth grade). Yet students have had experience with prime factors in Grade 6 and can intuitively understand that when the denominator (in simplest form) contains only prime factors of 2's and/or 5's, the decimal will terminate.

To convert $\frac{a}{b}$ (where a and b are integers) to a decimal we calculate $a \div b$ by long division. At each step in the long division algorithm, the remainder r is a whole number with $0 \leq r < b$. If $r = 0$, then the long division algorithm stops at that step, representing a terminating decimal. Otherwise r is a number from the following list: $1, 2, 3, \dots, b - 1$. By the b th step in the long division, some remainder has to appear again. When that happens, the process repeats, so that we start generating the same sequence of remainders that followed the first appearance of that remainder. And so on, the process continues to recycle forever, and we obtain a *repeating decimal*.

Let's follow this through for $6/11$. Start out by the division of 6 by 11. We get an approximation of .5 with a remainder of 5 hundredths. The next step is the division of 5 by 11, giving us a 4 with 6 left over, corresponding to an approximation of .54 with a remainder of 6 thousandths. But that leads us back to the division of 6 by 11, and so this two step process will now be repeated, over and over. We indicate this by writing

$$\frac{6}{11} = 0.5454545454 \dots$$

The abbreviated notation for the repeated part of the decimal is to put a line over the part that is repeated. Thus we can write this as

$$\frac{6}{11} = 0.\overline{54}.$$

In general, the decimal representation of a number is said to be repeating if it becomes periodic (repeating its values at regular intervals) and the infinitely-repeated portion is not zero. So, for example, $1/3$ has the decimal expansion $0.33333 \dots = 0.\overline{3}$, and the decimal $72.65431431431431 \dots$ can be expressed as $72.654\overline{31}$.

Let's consider $1/7$. If we calculate $1 \div 7$ we obtain the quotients 1,4,2,8,5,7 with corresponding remainders: 3, 2, 6, 4, 5, 1. Notice, each is greater than 0 and less than 7. Since there are only six possible remainders, the next step of the long division must produce one of these numbers. In fact, it is the first one, 3, so we conclude that $1/7 = 0.\overline{142857}$.

Try this with $26/111$. The first step in the long division produces a 0.2 with a remainder of 38, the second step gives 0.23 with a remainder of 44, and the third step a 0.234 with a remainder of 26. Since we started out with $26 \div 111$, this is our first repeater, and so the decimal expansion of $26/111$ is $0.\overline{234}$.

Section 1.3: Solve Percent Problems Including Discounts, Interest, Taxes, Tips, and Percent Increase or Decrease.

Solve real-world and mathematical problems involving the four operations with rational numbers. 7.NS.3

Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. For example: If a woman making \$25 an hour gets a 10% raise, she will make an additional $\frac{1}{10}$ of her salary an hour, or \$2.50, for a new salary of \$27.50. If you want to place a towel bar $9\frac{3}{4}$ inches long in the center of a door that is $27\frac{1}{2}$ inches wide, you will need to place the bar about 9 inches from each edge; this estimate can be used as a check on the exact computation. 7.EE.3

Another kind of fraction is the percentage. The word percent comes from the Latin phrase, *per centum*, literally “of one hundred” and is a contraction in English of the French *pour cent*. Long before decimals were used, the need to work with tenths and hundredths was evident. For example, in ancient Rome, the Emperor Augustus levied a tax on good sold at auction of 1 coin for every hundred coins of the transaction (and a tax of 4 coins per hundred for slaves). Since “per hundred” in latin is “per cent,” this is literally use of percentages. The percent sign was developed during the middle ages with the rise of banking, and comes from the shorthand of the Italian *per cento*.

The use of “percent” is an intuitive way, given our place value system, of speaking about parts of a whole, and are used in grades, sports, surveys and interest rates, just to name a few. For example, if 20% of the 7th grade plans to purchase Val-o-grams on Valentine’s Day next year, this tells us that 20 out of every 100 7th graders plan to purchase a Val-o-gram. Hence, 20% is the fraction $\frac{1}{5}$, written as a decimal 0.2 or 0.20.

Why use percents when we could use ordinary fractions? By using the denominator 100, it becomes easy to compare fractional amounts of different quantities. For example, if the fraction of 7th grade students at Pleasant Grove Junior High (PGJH) who like chocolate is $\frac{300}{400}$ and the fraction of 7th grade students at Oak Canyon Junior High (OCJH) who like chocolate is $\frac{315}{450}$, we have to do some calculating to tell which Junior High group of 7th graders has the greater fraction of students who like chocolate. On the other hand, if we are told that 75% of 7th grade students at PGJH like chocolate and 70% of 7th grade students at OCJH like chocolate, then we know immediately that 7th graders at PGJH have a greater fraction of students who like chocolate. So, percents are a way of putting fractions over a common denominator, making comparisons easy.

Since percents are alternative representations of fractions and decimals, it is important to be able to convert among all three forms.

Conversion 1: Percents to Fractions

Using the definition of percent, $25\% = \frac{25}{100}$

Conversion 2: Percents to Decimals

To convert a percent directly to a decimal, “drop the % symbol and move the number’s decimal point two places to the left.” This is because “percent” means “over 100”: 38% means $\frac{38}{100}$ which is represented by the decimal .38.

Conversion 3: Decimals to Percents

Here we merely reverse the shortcut of conversion 2, or we rewrite the decimal as an equivalent fraction. For example, $0.25 = \frac{25}{100}$ as a fraction with a denominator of 100, given that the “5” is in the hundredths place. We then utilize the definition of percent and we arrive at 25%. In essence, percents are obtained from the decimals by “moving the decimal point two places to the *right* and writing the % symbol on the right side.”

Conversion 4: Fractions to Percents

Percent is a fraction with a denominator of 100. Thus, we express $\frac{25}{100} = 25\%$, or $\frac{2}{5} = \frac{2 \cdot 20}{5 \cdot 20} = \frac{40}{100} = 40\%$, and $\frac{3}{25} = \frac{3 \cdot 4}{25 \cdot 4} = \frac{12}{100} = 12\%$.

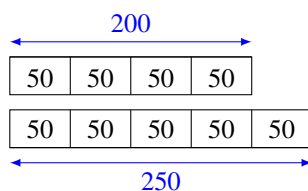
Conversion 4: Fractions to Decimals

This conversion was discussed in the preceding section.

EXAMPLE 8.

If a quantity grows, then the increase in the quantity, determined as a percent of the original, is the *percent increase*. If the number of students at Lincoln Middle School that participated in the Science Fair in 2011 was 200, but by 2012 had increased to 250 students, what is the percent increase in the number of students participating in the Science Fair?

SOLUTION. There were 50 more students who participated in 2012, that is, 50 represents 25% of the original 200 students; therefore, the percent increase was 25% as illustrated in the model.



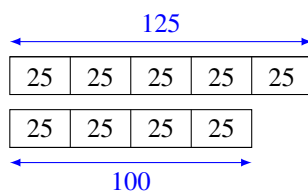
Additionally, the fraction change was an increase by $\frac{1}{4}$, the percent change was an increase by 25%, the 2012 fractional portion of the original would be represented as they have their 2011 students and lastly, the percent of the original would be represented as 125% of their 2011 students. Furthermore, the fraction change expression, and percent change expression would be illustrated as:

Fraction Change Expression	Percent Change Expression
$200(1) + 200\left(\frac{1}{4}\right)$ $= 200\left(\frac{5}{4}\right)$ $= 250$	$200(1) + 200(.25)$ $= 200(1.25)$ $= 250$

EXAMPLE 9.

If a quantity shrinks, then the decrease in the quantity, determined as a percent of the original, is the *percent decrease*. If the number of Grade 7 students at Lincoln Middle School that had land-line phones in 2011 was 125, but by 2012 had decreased to 100 land-line phones, what is the percent decrease in the number of land-line phones?

SOLUTION. There were 25 fewer land-line phones in 2012, that is, 25 represents 20% of the original 125 land-line phones. Therefore, the percent decrease was 20%, as illustrated in the model:



Additionally the fraction change was a decrease by $\frac{1}{5}$, the percent change was a decrease by 20%, the fractional portion of the original would be represented as there are $\frac{4}{5}$ the number of land-line phones as in 2011, and lastly, the percent of the original would be represented as 80% of their 2011 students. Furthermore, the fraction change expression, and percent change expression would be illustrated as;

Fraction Change Expression	Percent Change Expression
$125(1) - 125\left(\frac{1}{5}\right)$ $= 125\left(\frac{4}{5}\right)$ $= 100$	$125(1) - 125(.20)$ $= 125(.80)$ $= 100$

EXAMPLE 10.

Can percentage increase can be “reversed” by the same percentage decrease? Start with 100 and do a 10% increase. Follow this by a 10% decrease. Are you back at 100? Explain.

SOLUTION. 10% increase from the original 100 is an increase of 10, which equals 110. But a 10% reduction from 110 is a reduction of 11 (10% of 110 is 11), which means we ended up at 99 (not the 100 we started with). What happened? The 10% took us up 10, then 10% took us down 11, that is, the 10% increase was applied to 100, but the 10% decrease was applied to 110 as illustrated below.

