Appendix: Fluency

In sixth grade students bring together the arithmetic that they have learned over the preceding six years of education. It is the understanding of arithmetic procedures that provides the flexibility to extend the arithmetic propositions to other contexts (for example, linear functions in grade 8, polynomials in grade 9, and matrix algebra later on). The key to understanding arithmetic operations at a higher level is fluency in effecting those procedures. So we interpret *fluency* to mean more than automaticity. Students need to be able to move painlessly from exact computation to estimation of arithmetic results in context (for example: 95×32 is around $100 \times 30 = 3000$). Students need to be able to select the most efficient method of computation when needed (for example: $99 + 99 + 99 = 3(100 - 1) = 300 - 3 = 297$).

These talents can only result from the kind of confidence in knowledge that allows - even, encourages - exploration like these: a) by the time students leave middle school, they should be able to input any positive number in their calculator, and then punch the square root key over and over again to see what happens; b) alternate between punching in $1/x$ and the square root key repeatedly, and not be surprised by the result.

The standards (*6NS.2,3,4*) provide specific fluency objectives. The overall objective is this:

Compute fluently with multidigit numbers and find common factors and multiples,

where we take the word *fluently* as described above. The specific objectives are:

6.NS.2. Fluently divide multidigit numbers using the standard algorithm.

6.NS.3. Fluently add, subtract, multiply, and divide multidigit decimals using the standard algorithm for each operation.

6.NS.4. Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1 to 100 with a common factor as a multiple of a sum of two whole numbers with no common factor. For example, express $36 + 8$ *as* $4(9 + 2)$.

This appendix is divided into three sections, one for each of the stated objectives.The intent, in putting this material in an appendix is that it will not be covered as a unit, but will be used, where appropriate, and indeed maintained as a subtext throughout the grade. We leave it to the teacher to keep the idea of fluency in mind at all times, and to decide when to turn to this appendix for a class or two.

The 6.NS content standards on fluency significantly involve the standards of practice, for the whole concept of fluency has to do with the practice of mathematical skills and knowledge. The standards of practice that are specifically addressed in this appendix are:

- **Standard of Practice 5.** Use appropriate tools strategically.
  In particular, the calculator should be used: a) to complete precise, but routine, calculations for which the practitioner already has a sense of what to expect; b) to explore the consequences of hypotheses; c) to explore questions that arise out of natural inquisitiveness. Calculators should never be used to avoid

thinking; that is neither its role in science, nor its capability.

- **Standard of Practice 6.** Attend to precision.
  This standard is concerned mainly with precision in communication, but it also deals with precision in computation. Often, a student will relegate a routine, but tedious computation to the calculator. That is okay - that is the purpose of the tools we have. But the student must understand that asking the calculator to multiply two numbers produces a number correct to 12 significant figures. But the calculator cannot know if the numbers the operator input are the numbers the operator wanted to input. In this context *Attend to precision* means: is the 12 digit number provided by the calculator in the expected range? And only the operator can evaluate that, by having mentally worked out the expected range.

- **Standard of Practice 7.** Look for and make use of structure.
  This is probably the most significant of these standards for sixth grade. In this grade we stress understanding of the structure of arithmetic operations, so that the student can make effective use of that understanding, not just in sixth grade but throughout their education. In sixth grade we build an understanding of the structure of arithmetic so that they are willing to accept adaptation of that structure to new and broader contexts.

In view of 6.NS.3, we take the word "number" in 6.NS.2 to mean *whole number* and for *division* to mean: given numbers A and B, find numbers Q and R such that $A = QB + R$ and $R < B$. Note: if $A < B$, then $Q = 0$ and $R = A$, so in the sequel we will tacitly assume that we are dividing a given positive integer by a smaller positive integer.

We take 6.NS.3 to mean fluency in conversion between decimals and fractions as well: for example, multiplication by 3/4 is the same as multiplication by 0.75.

The first section is concerned with standard 6.NS.2. Attention here is restricted to division; assuming that fluency in the other arithmetic operations has been attended to in prior grades. However, since 6.NS.3 speaks of all the operations, and because of our focus on understanding the operations well enough to be able to mentally estimate a range for the answer, we here review all the operations on whole numbers, providing geometric interpretations that, together with place value, help to explain the algorithms.

In the second section we turn to standard 6.NS.4, as a natural sequel to the discussion of the arithmetic of whole numbers. This standard asks for fluency in the multiplication and division of small numbers, for a certain purpose: to find commonalities between two given numbers. Therefore, we concentrate on the factoring of whole numbers, starting with finding all factors, and then all prime factors. The prime factorization of a whole number leads easily to listing all factors, and to the discussion of commonalities between two numbers. An important point in finding the factors of a whole number A is that we need only divide A by primes up to $\sqrt{A}$. However, $\sqrt{A}$ is not in the sixth grade vocabulary. We avoid this problem by concentrating on the search for *factor pairs* (as discussed in fourth grade), and see that we can stop the search when we no longer get a new factor pair.

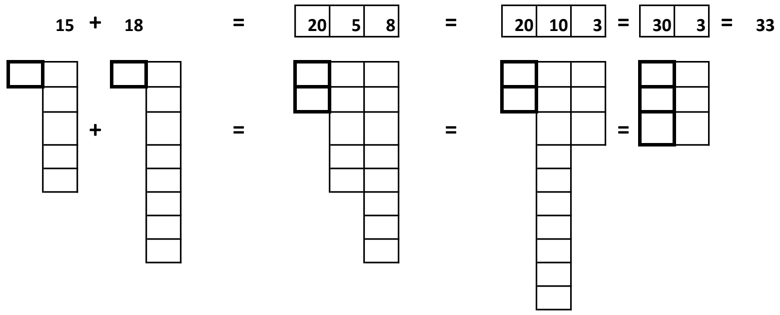
Finally, we return to 6.NS.3 which is about the operations on decimals, with attention to place value. Now, place values not only change to the left by increasing multiples of 10, but also change to the right by multiples of 1/10. If we had negative exponents at our disposal, it would help to explain the algorithms. Since negative numbers will be brought up later in sixth grade mathematics, we shall revisit this at that time. However, at first, our approach will be to replace an operation with decimal numbers by the same operation with whole numbers, by moving the decimal point (to the left or the right as necessary). In the end, we move the decimal point in the reverse direction to where it belongs.

Arithmetic Operations with Whole Numbers

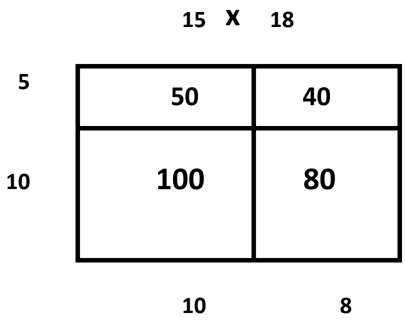
6.NS: Compute fluently with multidigit numbers, . . .

6.NS.2. Fluently divide multidigit numbers using the standard algorithm.

We start with the students' understanding of addition and multiplication: addition of two whole numbers corresponds to the joining of one group with the other; multiplication is repeated addition of a group to itself. The image below illustrates addition, using place value. In the rightmost image we see the idea of regrouping in action:



The figure to the right illustrates - in the context of place value - a fundamental arithmetic fact: the *distributive property* that relates multiplication and addition:



15X18 = 100+80+50+40 = 270

$$(A + B)C = AC + BC$$

and the more complicated version

$$(A+B)(C+D) = A(C+D)+B(C+D) = AC+AD+BC+BD\,.$$

With $A = 10, B = 5, C = 10, D = 8$ this becomes:

$$(10 + 5)(10 + 8) = 10 \times 10 + 10 \times 8 + 5 \times 10 + 5 \times 8 = 720$$

as shown in the figure to the right.

When we turn to the standard algorithms for addition and multiplication, we will see that that the driving force that makes those algorithms work is the distributive property, which when stated in generality says:

Distributive Property: If two expressions are to be multiplied, and each expression is a sum of terms, the result is the sum of all products of a term from the first expression and a term from the second expression.

Place Value. Now let us turn to the place value representation of whole numbers. We read a number like 523 as *five hundreds plus two tens plus three*, and so represents the algebraic sum $5 \cdot 100 + 2 \cdot 10 + 3 \cdot 1$.

Addition. When we add, say 97 to 523, we then go through the following procedure:

Write down the sum in explicit place value : $523 + 97 = (5 \cdot 100 + 2 \cdot 10 + 3 \cdot 1) + (9 \cdot 10 + 7 \cdot 1) =$

Remove parentheses and collect terms : $= 5 \cdot 100 + (2 \cdot 10 + 9 \cdot 10) + (3 \cdot 1 + 7 \cdot 1) =$

$$= 5 \cdot 100 + 11 \cdot 10 + 10 \cdot 1 = 5 \cdot 100 + (10 + 1) \cdot 10 + 1 \cdot 10 = (5 + 1) \cdot 100 + (1 + 1) \cdot 10 = 620\,.$$

The final line illustrates the process of *regrouping*. The fundamental principal is that a whole number is written as a sequence of digits (numbers between 0 and 9), where the place indicates a power of ten. Reading from the right, the first digit is the number of units (multiples of 10^0), the place to the left is the number of tens (multiples of 10^1), the place to the left of that is the number of hundreds (multiples of 10^2), and going forward, each new digit represents the number of the next power of 10. We finish the algorithm by collecting the summands together in a sequence of digit multiples of powers of ten. The fundamental rule for this is

$$10 \cdot 1 = 10, \quad 10 \cdot 10 = 1 \cdot 10^2\,, \quad \ldots, \quad 10 \cdot 10^n = 10^{n+1}$$

The student has learned how to do this without being explicit, using a well developed shorthand (or just memory) at each step. The point in writing this down in explicit detail is to illustrate how the student is to understand these operations, rather than doing them by rote. It is that understanding that is the key to the fluency that allows appropriate flexibility in estimating the result. Let us illustrate by example.

Example 1. At 9:00 on election night, the ballot count is: candidate A: 47560; candidate B: 44127. But now the returns from District 10 are just in: 2316 votes in District 10 cast for candidate A, and 7387 for candidate B: . Who is now in the lead?

Solution. The question is "*who is now in the lead ?*," not "*by how many votes*." So, we say that A's new tally is about 47,500 + 2,300 = 49,800 votes, while that for B is about 44,000 + 7,000 = 51,000 votes, telling us that candidate B has taken the lead! Now the actual sums are done on a calculator, and found to be 49,876 and 51,514. Because we made the quick estimate, we feel confident that we input the correct numbers.

Example 2. To date, my orchard has produced 23,420 bushels of apples and 16,870 bushels of pears. A bushel of apples brings in $7.00 and a bushel of pears, $10. To sustain my orchard, I need to bring in $300,000. Assuming the ratio of apples to pears remains the same, about how many more bushels of apples and pears do I need to collect to sustain the orchard?

Solution. Precision is not the issue: we just need to exceed $300,000 in income. So, we approximate. So far, we have more than 23,000 bushels of apples, and a more than 16,000 bushels of pears. At $7 a bushel of apples and $10 a bushel of pears, my orchard has already earned more than $23,000 \times 7 + 16,000 \times 10$ dollars.

$$23 \times 7 = 20 \times 7 + 3 \times 7 = 140 + 21 = 161 \quad \text{and} \quad 16 \times 10 = 160 \,,$$

which comes to $161,000+ $160,000 = $321,000. We are over the sustenance requirement by $21,000.

Note. Since we needed to exceed a certain amount, we *rounded down* to the nearest thousand. That worked, so we were ok. If the rounding down produced an income much less the required amount, we'd have to do the strict multiplication to see how much more fruit to pick. It is worth remarking, that in order to get a good quick estimate, it is best to *round to the nearest* thousand. In this case we would have gone to 23,000 bushels of apples and 17,000 bushels of pears, and our estimate would give us $331,000. But we wouldn't be *sure* this was less than the actual amount and that we had exceeded our goal. For the record, the income produced by the fruit already picked is

$$23,420 \times 7 + 16,870 \times 10 = 164,290 + 168,870 = 333,160 \,,$$

so the result of *rounding to nearest thousand* is still less than the actual sum.

Example 3. Gianni, Sylvia and Lester are at the amusement park, and they want to ride the Loop-de-loop. Tickets are $5.50 each. Gianni has $4.50, Sylvia has $7.50 and Lester has $4.50. Can they pool their resources and all ride the Loop-de-loop?

Solution. In order for all to ride, they need 3 tickets at $5.50 apiece, or $3 \times 5.50 = 16.50$ dollars. Since we are discussing "fluency," we'd like to point out that, on the run, this multiplication is most easily computed by repeated addition:

$$3 \times 5.50 = 5.50 + 5.50 + 5.50 = 5 + 0.5 + 5 + 0.5 + 5 + 0.5 = (5 + 5 + 5) + (0.5 + 0.5 + 0.5) = 16.50 \,.$$

All together they have, in dollars:

$$4.50 + 7.50 + 4.50 = 4 + 8 + 4.50 = 16.50 \,,$$

exactly what they need to buy 3 tickets,

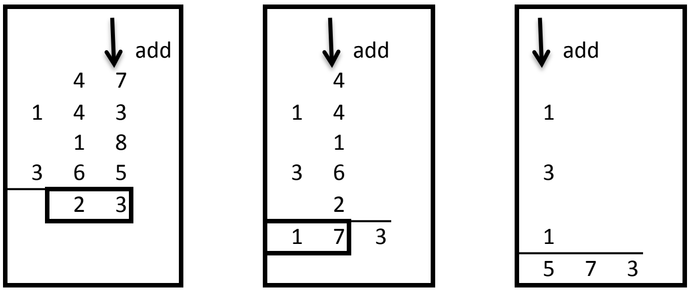
Before proceeding to other operations, let's look at how a column of numbers are added.

Example 4. 47 + 143 + 18 + 365 =?

Solution. First we put the summands in a column, as in the figure to the right.

	4	7
1	4	3
	1	8
3	6	5

Next we add the digits in the right most column (the ones (10^0) place), as in leftmost column below (getting 23). Keep the ones figure (3), and put the 2 in the tens (10^1) column., just above the "add" line. Now add the sum of the figures in the tens column (as in the second figure below) to get 17. Keep the 7 and put the 1 in the hundreds column, and finally add (as in the rightmost table) the figures in the 100s column. The result is 573.

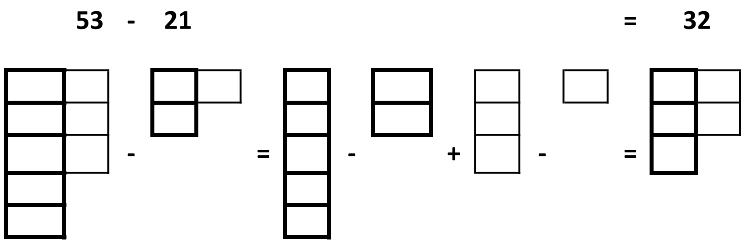


Subtraction. As with addition, we focus on the place value representation of number, so the actual subtraction we do is that between digits. Now, in any place the digit for the number to be subtracted may be greater than the digit in that place for the other number. In this case, we regroup *from* the place value just to the left. Note that, just as subtraction is the inverse of addition, the regrouping in subtraction is the inverse of that for addition. Let us illustrate:

Example 5. a) Subtract 21 from 53; b) Subtract 28 from 53; c) Subtract 46 from 201

Solution. a) To subtract 21 from 53, we write the numbers in place value, rearrange by place value and solve:

$$53 - 21 = (50 + 3) - (20 + 1) = (50 - 20) + (3 - 1) = 30 + 2 = 32 .$$



b) Now let's subtract 28 from 53, using the same ideas:

$$53 - 28 = (50 + 3) - (20 + 8) = (50 - 20) + (3 - 8) = 30 + (3 - 8) ,$$

whatever that last term (3-8) can mean. So, to give it meaning we *regroup* 30 + 3 to 20 + 13, and continue

$$30 + (3 - 8) = 20 + (13 - 8) = 20 + 5 = 25 .$$

c) 201 − 46 is a little harder, since there are no “tens” digits to regroup. But there are hundreds digits we can regroup into tens and rewrite:

$$201 - 46 = (2 \cdot 100 + 0 \cdot 10 + 1 \cdot 1) - (4 \cdot 10 + 6 \cdot 1) = (1 \cdot 100 + 10 \cdot 10 + 1 \cdot 1) - (4 \cdot 10 + 6 \cdot 1) =$$

and now we regroup the $10 \cdot 10 = 9 \cdot 10 + 10 \cdot 1$:

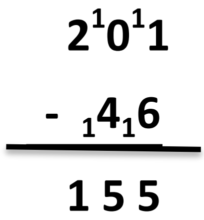
$$1 \cdot 100 + 9 \cdot 10 + (10 + 1) \cdot 1 - 4 \cdot 10 - 6 \cdot 1 = 100 + (90 - 40) + (11 - 6) = 100 + 50 + 5 = 155 .$$

An alternative to regrouping is that of adding a 10 in one place, and subtracting it from the next place. For example: in the last calculation we can get from

$$2 \cdot 100 + (0 \cdot 10 - 4 \cdot 10) + (1 - 6)$$

to viable subtractions by adding $10 \cdot 1$ and subtracting $1 \cdot 10$, and adding $10 \cdot 10$ and subtracting $1 \cdot 100$:

$$1 \cdot 100 + (10 \cdot 10 - 5 \cdot 10) + (11 - 6) = 100 + 50 + 5 = 155 .$$



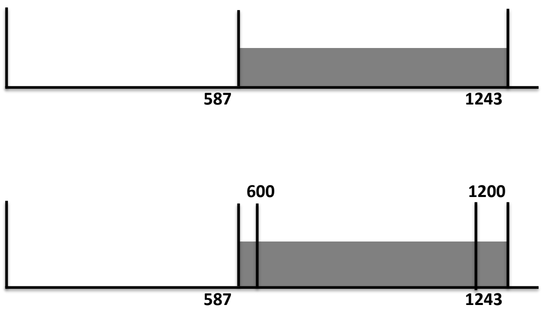
In practice, these calculations are made using placement to indicate the relevant power of zero, as in the image above.

Subtraction as Addition. If we draw a graphic of the subtraction of one whole number from a larger whole number, we can realize the operation as that of addition. This is particularly convenient for mental arithmetic.

Example 6. Subtract: 1243 − 587.

Solution. First, imagine the operation to be performed along a horizontal line, as in the first image below. Our object is to find the length of the grey section. We reduce this to simple additions as shown in the second image. Draw the lines at 1200 and 600. The piece of the grey section between these two lines is 600. The grey part to the left of this piece has length 13, and that to the right, 43. Thus

$$1243 - 587 = 13 + 600 + 43 = 656 .$$



1243 - 587 = 13 + 600 + 43

Multiplication. Once we understand the role the distributive property plays in the multiplication algorithm for place-value numbers, we can see the basics of the algorithm. To illustrate:

Example 7. Calculate 532×69.

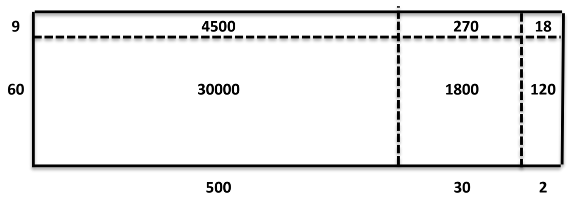
Solution. First, estimate the result, as a check on our calculation (whether by hand or calculator). Rounding to the nearest one-digit numbers gives us 500×70, which is 35000.

Now let's do the calculation by hand. Expand the numbers explicitly in place value and then take the sum of all the products of one term from the first factor by a term from the second factor:

$$532 \times 69 = (500 + 30 + 2)(60 + 9) = (500 \cdot 60) + (500 \cdot 9) + (30 \cdot 60) + (30 \cdot 9) + (2 \cdot 60) + (2 \cdot 9) =$$

$$30000 + 4500 + 1800 + 270 = 120 + 18 = 36,708\ .$$

Here is a geometric illustration of the use of the distributive property in this problem.



Of course, the point of the multiplication algorithm is to use place value to simplify this formula. So, we systematically multiply starting from the right. In the figure below, place value is indicated by the boxes. On the left, we transcribe the information given by the above, and on the right we illustrate the algorithm as taught in grade 4.

		5	3	2
			6	9
(1)		45	27	18
(2)	30	18	12	
(3)	30	63	39	18
(4)	30	63	40	8
(5)	30	67	0	8
(6)	36	7	0	8

		5	3	2
			6	9
(1)	4	7	8	8
(2)	31	9	2	
(3)	36	7	0	8

In both tables we see the two terms of the product on the top two lines. Each table executes the same algorithm: the table on the left makes the steps explicit, while the table on the right illustrates the standard algorithm in standard practice.

On the left, in line (1) we see the products of 9 with each place value of the first term of the product. In line (2) we have the product of 6 with each place value of the first term of the product. The terms are shifted over one place because the 6 represents $6 \cdot 10$. In line (3) we have summed the preceding two rows, so this line represents the product as $30 \times 1000 + 63 \times 100 + 39 \times 10 + 18 \times 1$. The rest of the lines show the regrouping to get the place value representation of the sum. The table on the right shows the same calculation, but with the automatic "regrouping" hidden.

In the real world, such a computation will be done on a calculator - but the calculator user needs some mental check that the result of the mechanical calculation is in the right ball park. To do that, we round to the nearest single digit number, as described at the beginning of this problem:, getting $500 \times 70 = 35,000$. Since the calculator shows 36,708, we can have confidence that that calculation is correct. Notice that the leading terms multiply whereas the number of following zeroes add. So, to estimate 532×69, we multiply $5 \times 7 = 35$, and follow it by 3 zeroes (since the 5 is followed by two places, and the 6 by one). Let us illustrate this.

Example 8. Estimate the following products:

a) 847×632 b) $6,017 \times 47$ c) $12 \times 51,000$ d) $143,000 \times 8$.

Solution. a) The leading digits are 8 and 6, and they are each followed by 2 places. So the product estimate is 48 followed by 4 zeroes: 480,000. We also know that this estimate is low, since we rounded down in both factors. The calculator gives 535,304, so we probably did the calculation correctly.

b) The best estimates for the leading digits are 6 and 5, since 47 is close 50. The 6 is followed by three places, and the 5 by one. So the product estimate is 30 followed by 4 zeroes: 300,000. The calculator gives 282,799. Note that the actual value is lower than the estimate; that is because we estimated 47 by the higher number 50. In general, when rounding one number up and the other number down, the estimate could be lower or higher than the actual value: what is important is that it is close.

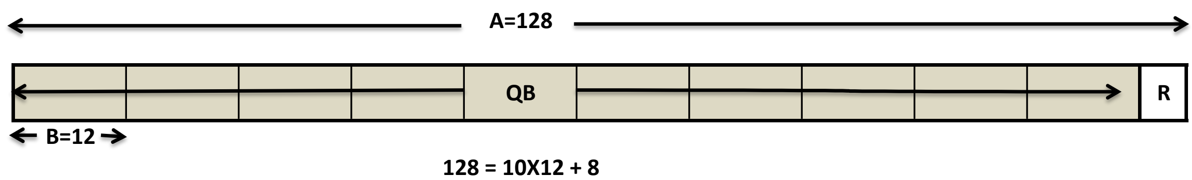
c) To estimate this, we don't have to work at all: 10 is close to 12, and since "multiplication by 10" amounts to putting another zero at the end, we immediately get the estimate 510,000. The actual multiplication also isn't very hard: $12 \times 51,000 = [(10 + 2) \times 51] \times 1000 = [510 + 102] \times 1000 = 612,000$.

d) Again, the estimate is easy: estimate 8 by 10, and conclude that the product is about, and less than 1,430,000. The actual product is 1,144,000.

We want to point out that, in multiplying two numbers on the calculator, it is easy to lose track of the number of places the result should have. So, if we multiply 113,000,000 by 704, when we punch in the first number, we may not punch in the right number of zeros. Estimating helps to check whether or not our input was correct. The answer has to be close to 7 followed by 10 zero place values. So, just counting place values can help us decide whether or not we punched in the right number of zeroes. Alternatively, we can just multiply 113 by 704, and remember that the result has to be followed by 6 zeros. Here we get $113 \times 704 = 79,552$, so our answer is 79,552,000,000.

Division. No wonder that students have difficulty with the division algorithm - it is a complex algorithm, and very difficult to write down in a sequence of determinate steps. It is also undesirable, for the computer code for this algorithm is easy for the computer to follow, but comprehensible only to the code writer. So, instead of trying to do this, we illustrate its use through examples. To begin with, we state the objective of this algorithm. The division algorithm for whole numbers says:

> **Division Theorem**. Given two whole numbers A and B, with $A \geq B$, there are whole numbers Q and R with $R < B$ such that $A = QB + R$. Q and R are uniquely determined by these conditions.



The graphic illustrates the theorem with $A = 128$, $B = 12$. Some language is desirable: in this theorem, A is called the *dividend*, B is the *divisor*, Q is the *quotient* and R is the *remainder*.

Example 9. Demonstrate the Division Theorem with $A = 35$ and

a) $B = 2$ b) $B = 5$ c) $B = 9$ d) $B = 11$ e) $B = 14$.

Solution.

- a) 2 divides 35 17 times with a remainder of 1. So $Q = 17$ and $R = 1$, and the statement of the division theorem is $35 = 17 \cdot 2 + 1$.
- b) 5 divides 35 7 times with no remainder. So, here we have $Q = 7, R = 0$, and $35 = 7 \cdot 5 + 0$.
- c) Here we may have to count the multiples of 9: 9, 18, 27, 36. Since 35 is between the third number and the fourth, we conclude that $Q = 3, R = 8$, and $35 = 3 \cdot 9 + 8$.

- d) Same here: Three elevens (3×11) gives 33 which is 2 less than 35, so $35 = 3 \cdot 11 + 2$, and we have a quotient of 3 and a remainder of 2.

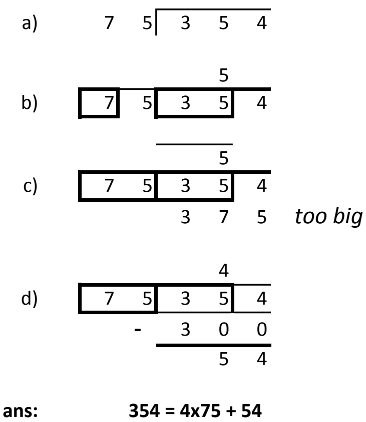
- e) The multiples of 14 are $14, 28, 42, \ldots$, so now $Q = 2$, and since $35 - 2 \cdot 14 = 7$, we have a quotient of 2 and a remainder of 7.

The operation called *the division algorithm* is a systematic technique to solve the problem: given A and B with $A \geq B$, find Q and R such that $A = QB + R$ with $R < B$. The purpose of Example 8 was to illustrate in simple contexts the way the division algorithm works. Basically it is this: make a *good* guess q for Q and let $R' = A - qB$. If $R' < B$, you're done. If not, apply the same technique to R', and continue the process until we end up with an R that is less than B.

Of course, the crux of this description is the phrase *good guess*; when do we know we've made a good guess? The answer is that that we guess one place value at a time. Starting with the top place values (a for A and b for the top one or two place values for B), we solve $a = qb + r$ and then calculate qB, subtract that from A to get $R = A - qB$. We've now found the top place value of the quotient, to proceed we now go through the same procedure with R replacing B. And so forth. This description is still too vague, but the best way to clarify it is to go through examples.

Example 10: $354 \div 75 = ?$

Solution. Let's follow the graphic to the right. In a) we have set the problem up for long division. We concentrate on the highest place value of both divisor and dividend. b) Since 7 is not less than 3, we look at $35 \div 7 = 5$ and try 5 as the leading place value of the quotient. In c) we have multiplied 75 by 5 to get 375 - telling us that 5 is too big. So, in d) we go with 4, calculate $4 \times 75 = 300$, and subtract from 354 to get 54, a number less than 75. So our division is complete: the quotient is 4 and the remainder is 54.



Example 11: Hercule's class has 23 students. His class is going to make a field trip to the state capitol which will will cost \$32 per student. The class has a fund raiser that produces \$654. Is that enough? If not, how much more has to be raised?

Solution. For a quick calculation, multiply 23 by 30 ($2 \times 23 = 46$, $3 \times 23 = 69$, so $23 \times 30 = 690$, which is more the the amount Hercule's class has, and less than 23×32, the amount needed. Now, since we know that $23 \times 30 = 690$, we need to just add $23 \times 2 = 46$ to 690, because

$$23 \times 32 = 23 \times (30 + 2) = 23 \times 30 + 23 \times 2 = 690 + 46 = 736);$$

so we need \$736 for all to make the trip.That is $46 + 36 = 82$ dollars more than the class has.

However, since we are talking about long division, let's complete the division. In line a) we've written the problem in computational form, and in line b) we've found that 3 as the first place value of the quotient is too big. So, in line c) we try 2, and find that $65 - 46 = 19$, which is less than 23, and we go to the next step: bring in the next place value (4): so what we now need to do is to find $194 \div 23$, in the same way. That is we look first at $19 \div 2$, which suggests a next place value of 8. The second line of d) is $8 \times 23 = 184$, so we have arrived at the answer. In terms of this problem, we have enough funds to send 28 students to the state capitol, with \$10 left over. That means that we need to raise another 4×32 less 10.

a)	2	3	6	5	4	
				3		
b)	2	3	6	5	4	
			6	9		*too big*
				2	8	
c)	2	3	6	5	4	
			4	6		
d)			1	9	4	
			1	8	4	
				1	0	

$$654 = 28 \times 23 + 10$$

Alternative method. To summarize: the idea behind the division algorithm is to find the best quotient for the leading place values of the divisor and dividend, then take the product of this best quotient with the divisor, subtract from the dividend and repeat. But why work so hard to find the *best guess*. Why not try any guess, so long as it gives us a product (with the divisor) that is less than the dividend. In this case, we have to keep track of place value. To illustrate:

Example 13: What is $899 \div 28$?

Solution. On the right we have indicated the intended long division, and alongside we've drawn two vertical signs, representing the tens place and the one place (a quick estimate, like $900 \div 30$ tells us to anticipate a two place quotient. Now look at $89 \div 28$, and guess a quotient of 2. Put 2 in the tens place to the right; then multiply 23×2 and put that under the 89. Subtract to get 33. That is larger than 23 (but not by much), so assume a quotient of 1 and repeat the process. Notice, we are still in the tens place, so that is where the 1 goes, just beneath the 2. Subtract 28 from 22 to get 5 (in the tens place, and then bring down the 9. We now want the quotient of 59 by 28; we already know 28×2, so we use that guess. But now we are in the ones column, so we put the 2 to the right of the first line. Subtract 56 from 59 to get the remainder 3, which is placed to the right of the second vertical bar. Now add along the answer columns to get 32 with a remainder of 3. That is $899 \div 28 = 32$ with a remainder of 3.

2	**8**	**8**	**9**	**9**			
		5	**6**		**2**		
		3	**3**				
		2	**8**		**1**		
			5	**9**		**2**	
			5	**6**			
				3			**3**
					3	**2**	**3**

Example 13: Let's try something harder: divide 72381 by 114.

Solution. In line a) we've written the division to be performed. Now, pick a multiple of 114 that is less than 732: let's try 3. $3 \times 114 = 342$, so in the next line, b), we write 34200 since the 342 goes in the highest three place values, leaving two to fill in. To the right of that write 300, for that is, so far, our quotient. c) Now subtract line b) from line a) to get the first remainder of 39081. d) from what we've done so far, it is clear that we want to put in 34200, which is 300×114. So we see we have another

a)	1	1	4	7	3	2	8	1				
b)				3	4	2	0	0		3	0	0
c)				3	9	0	8	1				
d)				3	4	2	0	0		3	0	0
e)					4	8	8	1				
f)					3	4	2	0			3	0
g)					1	4	6	1				
h)					1	1	4	0			1	0
i)						3	2	1				
j)						2	2	8				2
						0	9	3		6	4	2

 73281 = 642x114 + 93

summand of 300 to our developing quotient. e) Subtracting line d) from line c) we get a new remainder of 4881. f) Repeating the process, from what we've done we again want to use $3 \times 114 = 342$, so we put 3420 under 4881, and put a 30 to the right of that (the last 0 is put in because now there is only one place needing a value). g) This is the new remainder, the difference between the two lines. h) 114 is the biggest multiple of 114 that is less than 146, so we write in 1140, with a 10 to its right, signifying the new addend we have for the quotient. i) This is the difference between the preceding two lines, and j) is 2×114, the multiple of 114 less than 321. To the right we put a 2: the last addend of the quotient, for the new remainder (last line, 93) is less than 114. To the right of this last remainder we put the sum of all the quotient addends: 642, and conclude that $73281 = 642 \times 114 + 93$

Divisibility

There are some observations to be made that make finding digit factors of whole numbers relatively easy. For example, we know that any number, no matter how many places there are, is even if the digit in the one's place (10^0 position) is even; that is, one of $0, 2, 4, 6, 8$. To see why this is, we return to place value.

Let's start with an example: $2,458$. This is an even number because 8 is even. But why? Because

$$2,458 = 245 \times 10 + 8 .$$

Now since 10 is even, the first term is divisible by 2, and since 8 is also even, the sum, $2,458$ is even. In fact:

$$2,458 \div 2 = 245 \times (10 \div 2) + 8 \div 2 = 245 \times 5 + 4 = 1225 + 4 = 1229 .$$

This argument carries forth for any number no matter how many digits:

$$3,,578,647 = 357,864 \times 10 + 7$$

is odd because the first term is even, but the second, 7, is odd - so the sum is odd.

This same trick works for divisilbilty by 5. Writing $2,458 = 245 \times 10 + 8$, since 10 is divisible by 5, the first summand is divisible by 5, but the second, 8, is not, so the sum is not divisible by 5. In fact, since the only digits divisible by 5 are 0 and 5, we can conclude that a whole number is divisible by 5 precisely when it ends in a 0 or a 5.

Similarly, a whole number is divisible by 10 precisely when it ends in a zero (in fact this follows directly from the concept of place value).

Since 1, 2, 5 and 10 are the only factors of 10, this argument will not work for any other digit. However, a slight generalization works for 4: Since 100 is divisible by 4, we need to look only a the lowest two digits of a number to see if it is divisible by 4:

$$2,358 = 23 \times 100 + 58 .$$

Since the first term is divisible by 4, we need only ask if 58 is divisible by 4. It isn't, so 4 is not a factor of 2,358 (but is of 2,348, since $48 \div 4 = 12$).

A similar argument works for divisibility by 8: Since 8 divides 1000), we have only to look at the the last three digits. As an example, let's take 46,348. Since the last digit is 8, this is divisible by 2. Now we look at $46348 \div 2 = 23174$; since the last digit is even, we know that 23174 is divisible by 2, so $46348 = 23174 \times 2$. But nw, since 74 is not divisible by 4, the evenness ends here, and 46248 is not divisible by 8. Or we could have continued the division by 2: $46348 = 23174 \times 2 = 11587 \times 4$, and 11587 is *not* divisible by 2. Another way to look at it is this: $8 = 4 \times 2$, so if a whole number fails the 4-test, it isn't divisible by 8. If it passes the 4-test and the quotient ends up in a 2,4,6,8, then it is divisible by 8; otherwise not.

For 3 and 9 there is a neat little trick. Note that

$$10 = 9 + 1 , \quad 100 = 99 + 1 , \quad 1,000 = 999 + 1 , \quad 10,000 = 9,999 + 1 \text{ , etc. .}$$

Every positive power of 10 is the sum of a number whose only digit is 9: a 1 followed by k 0's is equal to k 9's plus 1. A number whose only digit is 9 is divisible by both 3 and 9. It follows that, for any number, if the sum of its digits is divisible by 3 or 9, the number is divisible by 9. Let's illustrate:

$3,567 = 3\times10^3+5\times10^2+6\times10+7 = 3\times999+3+5\times99+5+6\times9+6+7 =$ something divisible by $3 plus 3+5+6+7$.

Since 3 + 5 + 6 + 7 = 21, we can conclude that 3,567 is divisible by 3. But 21 is not divisible by 9, so 3,567 is not divisible by 9.

The remaining digit is 7: We could examine place value very carefully and come up with a protocol for determining divisibility by 7, but the protocol is far more difficult to execute than actual division. So, we do not state a rule.

In summary,

Tests for Divisibility

a) Divisibility by 2: The number ends in a 0, 2, 4, 6, 8.

b) Divisibility by 3: The sum of the digits of the number is divisible by 3.

c) Divisibility by 4: The number given by last two digits of the number is divisible by 4.

d) Divisibility by 5: The last digit of the number is 0 or 5.

e) DIvisibility by 6: The last digit is 0, 2, 4, 6, 8 and the sum of the digits is divisible by 3.

f) Divisibility by 7: No trick beats the division algorithm

g) Divisibility by 8: or any power of 2: divide by 2 consecutively until the last digit is odd. If you made k successive divisions, the number is divisbile by 2^k.

d) Divisibility by 9: The sum of the digits of the number is divisible by 9.

Example 14. Test each of the following numbers for divisibility by a positive digit.

a) 3,467,002 b) 77,760 c) 33,000,333 d) 40,446.

SOLUTION. a) The last digit is 2, so the number is divisible by 2 but not by 5 or 10. Divide again by 2: the last digit is odd. So, this number is divisible by 2, but not by 4. The sum of digits is 22, so the number is not divisible by 3 or 9. Being not divisible by 9, it is not divisible by 6. Finally, divide by 7 and get a whole number. Thus the digits that divide 3,467,002 are 2 and 7.

b) Since last digit is a zero, 77760 is divisible by 10, and therefore, divisible by 2 and 5. The last two digits are 60, which are divisible by 4, leaving a quotient of 19420 (which is even so 77760 is also divisible by 8). The sum of the digits is 27 so the number 77760 is divisible by 3 and 9. Because it is divisible by both 2 and 3, 77760 is divisible by 6. Now the first three digits are divisible by 7 (that is 77700is divisble by 7), but 60 is not, so our number does not have 7 as a factor. In summary, the basic divisors of 77760 are 2, 3, 4, 5, 6, 8, 9, and 10.

c) This number ends in 3, so is not divisible by 2,5 or 10, and by extension, by neither 4 or 6. The sum of the digits is 15, so it is divisible by 3 (but *not* 9), giving a quotient of 11,000,111 which (after trying to divide by 7) has no digit factors other than 3.

d) This number ends in a 6, so is divisible by 2, but not by 5. The quotient by 2t ends in a three, so 40,446 is not divisible by 4 or 8. The sum of the digits is 18, so our number is also divisible by 3 and 9. Being divisible by 2 and 3, it is also divisible by 6. By calculation, the number is not divisible by 7. In summary, the divisors are 2,3,6,9.

Example 15. You are the senior member of a team of 9 realtors in an agricultural realty agency. When a sale is executed, all partners receive equal shares, and if there is a remainder, the remainder goes to you as senior partner. In the month of October last year, the following sales were made:

a) 14,017 acres   b) 77,760 acres   c)11,010 acres   d) 40,100 acres.

How many more acres did you accrue than did your partners, in the month of October.

SOLUTION. a) The sum the digits is 13. This is 9+4, so when the equal allotments are done, there are 4 acres left over: these are yours. b) The sum of the digits is 27, divisible by 9, so you receive no bonus. c) The sum of the digits is 3, so the number is not divisible by 9. The highest number less that 11,010 that is divisible by 9 is 11, 007. Thus you receive 3 acres more than do your partners. d) By the same reasoning, 40,095 acres can be evenly divided among the partners, so the senior member receives an additional 5 acres. In total, the senior partner has received $4 + 0 + 3 + 5 = 12$ acres more than the other partners.

Factors and Multiples of Whole Numbers

Students first started learning about factoring whole numbers in third grade. grade. Then in fourth grade students understand that factors come in pairs and can find all factor pairs of numbers up to 100. The use of factor pairs shows students that to find all factors of a number less than 100, they need only actually divide by single digits. In sixth grade, we develop this idea and strive for fluency in finding factors for all numbers up to 144, and multiples of all number up to 12.

6.NS.4. Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1 to 100 with a common factor as a multiple of a sum of two whole numbers with no common factor. For example, express $36 + 8$ *as* $4(9 + 2)$.

We feel that it is important for students to understand that *factor* and *multiple* enjoy a dual relationship (as in *3 is a factor of 18* and *18 is a multiple of 3*). For this reason we seek fluency in finding multiples of numbers up to 12 and factors of numbers up to 144.

When we see a multiplication of whole numbers, such as $a \cdot b = c$ we say that a and b are a *factor pair* for c. We say that a and b are *factors* of c. Note that since $1 \cdot c = c$, 1 is a factor of every whole number, and any whole number is a factor of itself. In fact, 1 and c are always a factor pair for c. To test if a is a factor of c, we divide c by a. If the remainder is zero, a is a factor of c, as is the quotient $b = c/a$ and a and b are a factor pair.

Example 16: Find all the factors of 84.

SOLUTION. To simplify the problem, we search for factor pairs. First, 1 and 84 form a factor pair. Since 84 is even, we get the factor pair 2 and 42. Now we divide 84 by 3 and get a remainder 0, giving us the factor pair 3 and 28. Continuing in this way we find the factor pairs

1	2	3	4	6	7	12	...
84	42	28	21	14	12	7	...

We have ended at the pair 12,7 because that is the same as the preceding pair, but in reverse order. Furthermore, continuing would only be the same as reading the list of pairs backwards and in reverse order. Thus, 84 has these 12 factors:

1   2   3   4   6   7   12   14   21   28   42   84

Example 17: Find all the factors of 132.

SOLUTION. We proceed in the same way, stopping once we've reached the turning point:

1	2	3	4	6	11
132	66	44	33	22	12

and stop there, because the next factor pair (12,11) just repeats the last pair in this list. So, the factors of 132 are

1 2 3 4 6 11 12 22 33 44 66 132

Recall that a whole number greater than 1 is called *prime* if it has only the one factor pair: 1 and the number itself. Primes are $2,3,5,7,11,13,\cdots$. To test if a number is prime we just have to show, as above, that there are no factor pairs other than 1 and the number itself.

Any number can be written as a product of primes (allowing repetitions). Looking back at the examples above: the prime factors of 84 are $2, 3, 7$, and indeed $84 = 2\cdot2\cdot3\cdot7$. The prime factors of 132 are 2,3,11 and $132 = 2\cdot2\cdot3\cdot11$.

This leads to another way of find all factors of a given number: find the prime factorization. Then any factor is a product of some of those primes. Let's repeat Example 15, using this method.

Example 18: Find all the factors of 84.

SOLUTION. 84 is even, so $84 = 2 \cdot 42$. 42 is even, so $84 = 2 \cdot 2 \cdot 21$. 21 has the factor pair 3,7, so $84 = 2 \cdot 2 \cdot 3 \cdot 7$, and we are done.

Common Factors and Multiples

Example 19. Find all numbers that are factors of *both* 84 and 132.

SOLUTION. We can just look at the lists in Examples 13 and 14 and see that the list of factors of both 84 and 132 is $\{1, 2, 3, 4, 6, 12\}$. Note that the greatest common factor of 84 and 132 is 6.

Example 20. Find the common factors of 75 and 100.

SOLUTION. Both numbers end in a 5 or 0, so 5 is a factor of both: $75 = 5 \cdot 15, 100 = 5 \cdot 20$. Now let's find the common factors of 15 and 20: again 5 dfivides both, so $15 = 5 \cdot 3, 20 = 5 \cdot 4$. Thus $5 \times 5 = 25$ is also a common factor of 75 and 100. There are no further common factors, since 3 and 4 have no common factors. Thus the factors common to 75 and 100 are 1,5,25, and 25 is the *greatest common factor*. Finally, we've also found the prime factors of both numbers: $75 = 3 \cdot 5 \cdot 5, 100 = 2 \cdot 2 \cdot 5 \cdot 5$.

The *greatest common factor (GCF)* of two numbers (that is, that common factor that is the largest) plays an important role in much of arithmetic. For example, if we want to divide two numbers A and B and happen to know the GCF, then we can cancel out the GCF and look at the quotient of the other factors.

Example 21. a) $100 \div 75 = ?$ b) $132 \div 84 = ?$

SOLUTION. a) We know that 25 is the GCF of 100 and 75: $100 = 4 \cdot 25, 75 = 3 \cdot 25$. Thus $100 \div 75 = 4 \div 3$. Since $4 \div 3$ has a quotient of 1 and a remainder of 1, the division theorem says: $4 = 1 \cdot 3 + 1$. Now multiply by the GCF, 25, and get the answer: $100 = 1 \cdot 75 + 25$, so $100 \div 75$ has a quotient of 1 and a remainder of 25.

b) Similarly, from Examples 14 and 15, we know that the GCF of 132 and 84 is 12, we know that $132\div84 = 11\div7$. Now, $11 = 1 \cdot 7 + 4$; multiplying by 12, we get $132 = 1 \cdot 84 + 48$.

At the other side of finding common factors of two numbers, is that of finding common multiples of two numbers. Of course, there are infinitely many: if C is a common multiple of A and B, then so are $2 \cdot C, 3 \cdot C, 4 \cdot C, \ldots$. But there will be a*least common multiple* and that is designated the LCM.

	80, 100	16, 132	24, 72	42, 72
GCF	$2 \cdot 2 \cdot 5$	$2 \cdot 2$	$2 \cdot 2 \cdot 2 \cdot 3$	$2 \cdot 3$
LCM	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5$	$2 \cdot 2 \cdot 2 \cdot 3 \cdot 11$	$2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$	$2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7$

Example 22. What is the LCM of 75 and 100?

Solution. One way is to take one of the numbers and find the first multiple of that number that is divisible by the other. So, look at 100, 200, 300, 400, . . . , and divide them successively by 75, until we have no remainder. $100 \div 75$ has a remainder of 25; $200 \div 75$ has a remainder of 50, and $300 \div 75$ has a remainder of 0. So, 300 is the LCM of 100 and 75.

There is an easier way. Since we know the prime factorizations : $75 = 3 \cdot 5 \cdot 5, 100 = 2 \cdot 2 \cdot 5 \cdot 5$, we deduce that a multiple of both 75 and 100 must have a prime factorization that includes all of those prime factors. To be a multiple of 75, it must include a 3 and two fives. To be a multiple of 100 it must include two twos and two fives. The smallest number that has those properties is $2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 = 300$.

Example 23. What is the LCM of 84 and 132?

Solution. We have the prime factorizations

$$84 = 2 \cdot 2 \cdot 3 \cdot 7\ , \quad 132 = 2 \cdot 2 \cdot 3 \cdot 11\ .$$

So the LCM is $2 \cdot 2 \cdot 3 \cdot 7 \cdot 11 = 924$.

Example 24. Find the GCF and LCM of the pair of numbers:

a) 80, 100 b) 16, 132 c) 24, 72 d) 42, 72

Solution. We have the prime factorizations:

$$a)\ 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5\ , \quad 2 \cdot 2 \cdot 5 \cdot 5\ , \quad b)\ 2 \cdot 2 \cdot 2 \cdot 2\ , \quad 2 \cdot 2 \cdot 3 \cdot 11\ ,$$

$$c)\ 2 \cdot 2 \cdot 2 \cdot 3\ , \quad 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3\ , \quad d)\ 2 \cdot 3 \cdot 7\ , \quad 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3\ .$$

Example 25. Add the fractions:

a) $\frac{1}{3}, \frac{1}{9}$ b) $\frac{3}{4}, \frac{7}{10}$ c) $\frac{7}{6}, \frac{8}{9}$ d) $\frac{11}{42}, \frac{31}{72}$

Solution. In each of these problems, we want to rewrite the fraction so as to have the same denominator. So, since 1/3 and 3/9 represent the same number, and 3/9 and 1/9 have the same denominator we add as follows:

a)
$$\frac{1}{3} + \frac{1}{9} = \frac{3}{9} + \frac{1}{9} = \frac{4}{9}$$

b) Look for a common denominator: that is the same as looking for a common multiple of 4 and 10. Clearly the product, 40 , will do. So

$$\frac{3}{4} + \frac{7}{10} = \frac{30}{40} + \frac{28}{40} = \frac{58}{40}\ .$$

Even though this is not in lowest term, the expression tells us a lot; for example, that $\frac{3}{4} + \frac{7}{10}$ is almost $1\frac{1}{2}$.

c) Addition of disparate fractions requires finding a common multiple of the denominators; the product is always a candidate. But, as this case shows, the product of the denominators is 54, but the least common multiple is 18, a much more manageable number. So, compare the following computations:

$$\frac{7}{6}+\frac{8}{9}=\frac{63}{54}+\frac{48}{54}=\frac{63+48}{54}=\frac{111}{54}=\frac{37}{18}\ ;$$

$$\frac{7}{6}+\frac{8}{9}=\frac{21}{18}+\frac{16}{18}=\frac{21+16}{54}=\frac{37}{18}\ ;$$

d) If you did well on the preceding, you have graduated to the calculator for this one. Nevertheless, we want a quick check that we calculated correctly: the first fraction is a little larger than 1/4, and the second, a little less than 1/2. So the result should be near (1/4+1/2) = 3/4. The actual calculation shows the sum to be 0.69246026. . . .

When performing certain arithmetic computations, it often makes problems easier to be cognizant of factoring and arithmetic rules, particularly when working together. We will demonstrate how this works in a series of examples.

Example 26. a) $84 \div 36 = ?$ b) $24 \times 25 =?$ c) $\frac{28}{5} \times \frac{25}{7} =?$ d) $\frac{21}{4} \div \frac{24}{5} =?$.

SOLUTION. a) It should by now be standard to eliminate common factors, but we include this here to illustrate the interaction of factoring and arithmetic rules.

$$84 \div 36 = \frac{84}{36} = \frac{12\times 7}{12\times 3} = \frac{7}{3}\ ,$$

because the common factor 12 cancels.

b) Here we use commutativity of multiplication:

$$24 \times 25 = (2 \times 12) \times (5 \times 5) = (2 \times 5) \times (12 \times 5) = 10 \times 60 = 600\ .$$

c)

$$\frac{28}{5}\times\frac{25}{7}=\frac{28}{7}\times\frac{25}{5}=4\times 5=20\ .$$

d)

$$\frac{21}{4}\div\frac{24}{5}=\frac{21}{4}\times\frac{5}{24}=\frac{21}{24}\times\frac{5}{4}=\frac{7}{8}\times\frac{5}{4}=\frac{35}{32}\ .$$

In summary: operations involving multiplication and division is simplified by cancellation of common factors. in problems involving addition and subtraction, identification of common factors combined with the distributive law can effect simplifications.

Example 27. a) $48 + 36 =?$ b) $28 + 52 =?$ c)

SOLUTION. a) Since 12 is a common factor of the summands, we can factor it out; what remains is a simpler computation:

$$48 + 36 = 12 \times 4 + 12 \times 3 = 12 \times (4 + 3) = 12 \times 7 = 84\ .$$

b) Similarly

$$28 + 52 = 4 \times 7 + 4 \times 13 = 4 \times 20 = 80\ .$$

c)

Example 28. When Troy and Chrissie got married the put together their farm holdings. Troy had six 48-acre plots and Chrissie owned four 18 acre plots. How much acreage do they have together?

SOLUTION. The combined acreage is $6 \times 48 + 4 \times 18$. Now, think factors:

$$6 \times 48 + 4 \times 18 = 6 \times 48 + 6 \times 12 = 6 \times (48 + 12) = 6 \times 60 = 360\ .$$

Just as common divisors are useful in division of whole numbers, common multiples are necessary in addition of fractions. Recall the number line interpretation of fractions: for a positive integer a, $1/a$ refers to one of a

equal parts of the unit length, and, for a whole number b, b/a consists of b repetitions of $1/a$. That leads to an understanding of the addition of two fractions with the same denominator: b copies of $1/a$ plus c copies of $1/a$ gives $b + c$ copies of $1/a$. Algebraically this is represented by the distributive property:

$$\frac{b}{a} + \frac{c}{a} = b \cdot \frac{1}{a} + c \cdot \frac{1}{a} = (b + c) \cdot \frac{1}{a} = \frac{b + c}{a} .$$

Now, let's recall how we add disparate fractions, through examples. Keep in mind that in adding fractions we need to find a *common denominator*, and need not fuss for the least common denominator.

Arithmetic Operations with Decimals

6.NS.3. Fluently add, subtract, multiply, and divide multidigit decimals using the standard algorithm for each operation.

Addition and Subtraction. The algorithms for the arithmetic operations is the same for multidigit decimals as the algorithm for whole numbers, with one additional ingredient: the placement of the decimal point in the answer. For addition and subtraction this is a simple task: it goes in the same position as that for the given numbers. If the numbers have different size decimal parts, we make them the same by filling in the empty spaces with zeroes. Here we illustrate the possibilities:

Example 29. Add or subtract as indicated:

a) $3.54 + 7.28$ b) $1.89 + 3.6$ c) $7.28 - 2.54$ d) $3.6 - 1.89$

Solution.

a)	**3. 5 4**	**b)**	**1. 8 9**	**c)**	**7. 2 8**	**d)**	**3. 6 0**
	7. 2 8		**3. 6 0**	**-**	**3. 5 4**		**1. 8 9**
	10. 8 2		**5. 4 9**		**3. 7 4**		**1. 7 1**

We have not made explicit the regrouping that has taken place, for by this time, students should be quite able to see where it occurs.

Example 30. Siegesmund is making three berry pies for the class picnic. The recipe calls for a cup each of blueberries, raspberries and currants. He goes to the store with \$14.35. He finds that the price of a cup of blueberries is \$1.57, for a cup of raspberries, \$2.09 and for currants \$.89.

- a) How much will he spend on the berries?
- b) He remembers that his girlfriend Melinda hates currants, so he decides change the recipe to: two cups of blueberries, 1 cup of raspberries and no currants. Now how much will he spend?
- c) In case a) how much change does he have after the purchase?
- d) In case b) how much change does he have after the purchase?

Solution. a) He will get 3 cups of blueberries at \$1.57/cup, 3 cups of raspberries at \$2.09/cup and 3 cups of currants at \$.89/cup. So he will spend:

$$3 \times (1.57 + 2.09 + 0.89) = 3 \times 4.55 = 13.65 .$$

b) Each pie now costs $2 \times 1.57 + 2.09 = 5.23$, so the berry cost for 3 pies is $3 \times 5.23 = 15.69$.

c) He came with \$14.35 and spent \$13.65, so he has $35 + 35 = 70$ cents change.

d) He came with \$14.35 and spends \$15.69, so has to borrow \$1.34 from the grocer.

e) OK, there is no part e), but I don't want to leave Siegesmund between a rock and a hard place. So, I suggest to him that he make one pie *special* for Melinda, and the other two according to the original recipe. A Melinda pie costs \$5.23 (in berries), and a regular pie, \$4.55. So, in this way he spends $5.23 + 2 \times 4.55 = 5.23 + 9.10 = 14.23$, and comes home with 12 cents, a happy girlfriend and a contented class!

Multiplication and Division. Here the rule is almost as simple, but more difficult to track. To multiply two decimals, we move the decimal place over so that both numbers are whole numbers. Now multiply those whole numbers and then put the decimal point over the sum of of the number of places moved for the two multipliers. For division, the algorithm is the same, but with the word *sum* replaced by *difference*. But we'll get to that; let's look at multiplication.

Example 31. Multiply: 36.42×0.75. SOLUTION. . We multiply 3642 by 75. In doing so, we have moved the decimal point 2 places for each number. So, in the result we have to put the decimal point in by 4 places.

$$3642 \times 75 = 273150 \quad \text{so} \quad 36.42 \times 0.75 = 27.315 .$$

We can see what is going on with a little bit of algebra:

$$36.42 \times 0.75 = 3642(10^{-2}) \times 75(10^{-2}) = (3642 \times 75)(10^{-2} \times 10^{-2}) = 273150 \times 120^{-4} = 26.315 .$$

Example 32. a) Multiply: 4.715×0.8. b) Multiply: 36.35×80.

SOLUTION. a) $4715 \times 8 = 37720$. To get to these whole numbers, we moved the decimal to the right 3 places in the first number, and one place in the second. So, the correct placement of the decimal point in the answer is 4 places to the left. The answer is 3.7720. We should always check that we've made the right calculation: eight-tenths of 4.715 has to be about 1/5 smaller than 4.7; so 3.7 something looks correct.

b) $3635 \times 8 = 29,080$. We moved the decimal point two places to the right in the first number, and one place to the left for the second number. That gives us a move of 1 place to the right; to correct that we move 1 place to the left in the answer to get 2,908.

Example 33. Nkotiti buys three and a quarter pounds of bananas at \$.69 per pound. How much did it cost Nkotiti?

SOLUTION. First, let's make an estimate: three pounds at \$.70 per pound comes to \$2.10; we take that as an estimate. Now, performing the multiplication:

$$3.25 \times 0.69 = (325 \cdot 69) \times 10^{-2} \cdot 10^{-2} = 22,425 \times 10^{-4} = 2.2425 ,$$

which the store will round up to \$2.25.

Here we concentrate on the placement of the decimal point: (34.7)x(1.05) is 347x105 with the decimal point placed 3 digits from the right. WHY? Remember that 6th grades have yet to see negative numbers. So how do we account for negative exponents?

Division. Let us start at the division theorem for whole numbers: For two whole numbers A and B there exist (unique) whole numbers Q and R such that $A = QB + R$ and $R < B$. We say that "A divided by B is equal to Q with remainder R. Another way to state the division thoerem is with fractions:

$$\frac{A}{B} = Q + \frac{R}{B} \quad \text{with} \quad \frac{R}{B} < 1 .$$

Now, if we want to express R/B as a decimal we just continue the long division by introducing decimals. For example, if we want to express $647 \div 5$ as a decimal, we replace 647 by 647.000, with as many zeros as we need for the desired degree of accuracy.

Example 34. Perform the indicated division, correct to 2 decimal places:

a) $647 \div 5$ b) $115.7 \div 5$ c) $115.7 \div 6$ d) $115.7 \div 15$.

SOLUTION.

a)

```
     1 2 9. 4
  5| 6 4 7. 0 0
     5 0 0.
     1 4 7.
     1 0 0.
       4 7.
       4 5.
         2. 0 0
         2. 0 0
```

On the left we display what the division looks like: just as in dividing whole numbers, but now we have included two place values to the right of the decimal point, indicating tenths place and hundredths place. The division proceeds as with whole numbers, while keeping track of the decimal point. Note that it remains in the same place throughout the division. Finally once we've calculated the digit for the tenths place, we have a remainder of zero. Thus we've finished and the answer is $647 \div 5 = 129.4$.

b)

```
       2 3. 1 4
  5| 1 1 5. 7 0
     1 0 0.
       1 5.
       1 5.
         0. 7
         0. 5
            2 0
            2 0
```

The only difference between b) and a) is that the dividend has a decimal part. We just put that in, add a 0 in the hundredths place, and proceed as above. Again we end up with a remainder of 0, so the result is precise: $115.7 \div 5 = 23.14$.

c)

```
       1 9. 2 8 3
  6| 1 1 5. 7 0 0
     6 0.
     5 5.
     5 4.
       1. 7
       1. 2
          5 0
          4 8
            2 0
            1 8
              2 0
```

The procedure for c) is once again the same, however, when we get to the second decimal place, we do not have a remainder of zero. Thus the result: $115.7 \div 6 = 19.28$ is correct to 2 decimal places. Should we continue the long division further, we will continue to obtain 3's for as long as we have patience to go.

d)

```
            7. 6 1 1
  1 5| 1 1 5. 7 0 0
       1 0 5.
         1 0. 7
           9. 0
           1. 7
           1. 5
              2 0
              1 5
                5 0
```

For d) the procedure remains the same, although the divisor now has two digits. We still proceed with the division algorithm for whole numbers, while attending to the position of the decimal point. We find that, up to two decimal places, $115.7 \div 15 = 7.61$. We should notice that if we continued the long division, we'd get 3's all the way down.

The most general situation along these lines is that of dividing a decimal number by another decimal number as in, say $115.7 \div 5.35$. If we multiply both numbers by the same number, the result of the division is the same. So, we can multiply both numbers by 10 as many times as we need to to make the divisor a whole number. Then we proceed as in Example 30. Restating this procedure algebraically:

$$115.7 \div 5.35 = \frac{115.7}{5.35} = \Big(\frac{115.7}{5.35}\Big)\Big(\frac{10 \cdot 10}{10 \cdot 10}\Big) = \frac{11,570}{535} = 11,570 \div 535\ .$$

Example 35. Perform the indicated division, correct to 2 decimal places:

a) $115.7 \div 5.35$ b) $463.26 \div 1.3$ c) $8500 \div 4.25$ d) $115.7 \div 0.07$.

SOLUTION. The issue here is the precise location of the decimal point. The actual division can be done on a calculator, but the operator must have an estimate of the result as a check on the calculation.

- a) $115.7 \div 5.35$ is about $100 \div 5 = 20$. Now let's do the actual division and see if the answer is near or about 20:
$$115.7 \div 5.35 = \frac{115.7}{5.35} = \frac{11,570}{535} = 21.63 .$$
- b) $463.26 \div 1.3$ is going to be about (in fact *more than*) a third less than 450, so about 300. Calculating:
$$463.26 \div 1.3 = \frac{463.26}{1.3} = \frac{4632.6}{13} = 356.35 .$$
- c) $8500 \div 4.25$ is going to be around $8000/4 = 2000$. In fact
$$8500 \div 4.25 = \frac{8500}{4.25} = \frac{850000}{425} = 2000 .$$
- d) $115.7 \div 0.07$ is going to be *of the order of* $100 \div \frac{1}{10} = 100 \times 10 = 1000$. Calculating:
$$115.7 \div 0.07 = \frac{115.7}{0.07} = \frac{11,570}{7} = 1652.86 .$$

Notes

- i) In performing arithmetic operations with decimals, we have the context in mind, for often the decimal is an approximation to the actual number. Let us illustrate:
- ii) When we convert fractions to decimals, we have to specify the degree of accuracy we want. So, if we convert 1/7 to a decimal on a calculator we get a response like 0.14285714 or 0.1428571428571428, depending upon the accuracy of our calculator. Neither decimal is precise; in fact if we asked for a thousand, or a million decimal places, we still would not have a precise decimal representations. But for most purposes, two to four decimal places is sufficient for calculation, depending upon the context
- iii) When working with dollars and cents, the decimals are precise: \$14.35 means fourteen dollars and thirty-five cents. But now suppose that this is a lunch bill and we want to add a 12% gratuity. We multiply \$14.35 by 1.12 to get \$16.072 which is precise, but not dollars and cents. So, we either *round up* to \$16.08 or *round down* to \$16.07. We note that a bank, in any transaction, will round up for income and round down for outgo. Otherwise they will lose (rather than gain) fractions of a cent. For one transaction the difference seems absurd, but the banks typically do tens of thousands of transactions a day.
- iv) Thinking of decimals as approximations of precise numbers, we have to be cognizant of the fact that operations among numbers magnify the error in the approximation. This is illustrated in the following problem.

Example 36. Express $\frac{2}{9} \times \frac{3}{11}$ as a decimal correct to three places.

SOLUTION. There are two ways to go.

- A. Multiply 2/9 by 3/11 to get 2/33, and calculate the long division $2 \div 33$ to three places to get 0.061.
- B. Express both 2/9 and 3/11 as numbers correct to three decimal places, and multiply those numbers. We get the multiplication $0.222 \times 0.272 = .060384$, which rounds to 0.060.

Which answer is correct (to three decimal places)? In case A we performed the multiplication with the precise fractions, and then converted to the three decimal approximation. In case B, we first did the decimal conversion, then multiplied and then converted to three decimals.

The point here is that fractions are precise, whereas the decimal representation is, in most cases, an approximation. So, a good rule to follow s to use precise expressions for numbers until the end of the computation, and *then* convert to decimal approximation. So A is the answer correct to three decimal places

Example 37. Seven young adults are going to a University dance. Tickets for the dance are \$17.50 a couple. Singles have to pay the full couple price. a) They decide to divide the total cost evenly. How much does each one pay? b) Suppose the seven actually consist of three couples and a single. In consideration of the single, they decide to share the cost of a fictitious eight person among them. Now how much is each share.

a) The total cost for the seven is that for four couples, so is $4 \times 17.50 = 70.00$ Dividing that into 7 shares comes to \$10.00/share.

b) One way to do this is: There are 3.5 couples, so each couple pays $17.50 \div 3.5 = 20,00$, so each couple pays \$20 and the single pays 10. Another way to do this is to calculate the cost per person: $17.50 \div 2 = 8.75$. So each person pays \$8.75 plus a fair share of the \$8.75 of the person who is not there. That is $8.75 \div 7 = 1.25$. Thus the cost per person is \$10.00, with the three couples paying \$20.00 and the single paying \$10.00.

c) We understand that the question has no part c), but it might be desirable to contemplate why each one of these calculations comes to \$10.00 per person.