

# Chapter 1

## Linear Equations in One Variable

The first three chapters of grade 8 form a unit that completes the discussion of linear equations started in 6th grade, and their solution by graphical and algebraic techniques. The emphasis during these three chapters moves gradually from that of “equations” and “unknowns” to that of “functions” and “variables.” The first chapter is about solving linear equations; in the second we move to the graphical interpretation of linear expressions and the understanding of the constant rate of change for linear functions (to be compared to the constant of proportionality of a proportional relation), and then to the “slope” of a line, and how it can be interpreted as the rate of change of the associated linear expression. Note that in this discussion we have been using the word *expression* rather than *function*, so that the students will become familiar with the idea of evaluating linear expressions, and graphing those values, as a lead in to the concept of function. This is engaged in the third chapter in which the entire subject of linear functions is brought together and examined from a variety of perspectives. In Chapter 4 we return to solutions of equations, this time with pairs of equations in two variables for which we seek values of the variables that solve both equations (the solutions of the simultaneous equations). Finally, in Chapter 5 our focus turns completely to the concept of function, shifting emphasis to describing the relation between the two variables, rather than the mechanics of the function.

This chapter begins with a focus on the distinction between expressions and equations. The analogy is with language: the analog of “sentence” is *equation* and that of “phrase” is *expression*. An equation is a specific kind of sentence: it expresses the equality between two expressions. These equations involve certain specific numbers and letters. We refer to the letters as *unknowns* - that is they represent actual numbers which are not yet made specific; indeed, the task is to find the values of the unknowns that make the equation true. If an equation is true for all possible numerical values of the unknowns (such as  $x + x = 2x$ ), then the equation is said to be an *equivalence*. Arithmetic operations transform expressions into equivalent expressions; we come to understand that for linear expressions, the converse is true: we can get from one expression to an equivalent one by a sequence of arithmetic operations.

In chapter 2 we begin to change the way we look at the letters used in algebra from that of *unknown* to *variable*, and together with that, the understanding of an equation involving two variables as expressing a relation between them. We do this first in the context of proportion, but then go to general linear equations and the ideas of *rate of change* and *slope* of the graph. But in chapter 1 we are only interested in finding that (or those) number(s), if any, which when substituted for the unknown make the equation true. These are the *solutions*. We do this in specific contexts, seeing how to translate language sentences about numbers into equations involving linear expressions.

### Linear Expressions

A *linear expression* is a formula consisting of a sum of terms of the form  $ax$  and  $b$ , where  $a$  and  $b$  are numbers and  $x$  represents an unknown. By *unknown* we mean a symbol which stands for a number; it could be a specific one yet to be determined, or one yet to be chosen, or any possible number, depending upon the context. Since “ $x$ ” represents an unknown, we could replace it by any letter and still have the same sentence. In particular, in solving

a problem in context, it is a good idea to pick a letter suggested by the context.

#### EXAMPLE 1.

These are all linear expressions:

$$\mathbf{a.} \ 3x - 5 \quad \mathbf{b.} \ 3t \quad \mathbf{c.} \ -5 \quad \mathbf{d.} \ 3u + 2u + 17 \quad \mathbf{e.} \ 3x - \frac{12x - 16}{4} \quad \mathbf{f.} \ \frac{2}{5}y + \frac{3}{10} \quad \mathbf{g.} \ 6(2x - 5) + 11$$

Notice (as in examples **b.** and **c.**) that we could have either  $a$  or  $b$ , or both, equal to zero. When we have a particular problem, we are unlikely to start with a linear expression of the form **c.**, but, in our manipulations with the expression, we may end up there. For example, when we combine terms in **e** we end up with  $3x - 3x + 4$ , which is just 4.

To *evaluate* a linear expression is to substitute a number for the unknown and calculate the resulting value. For example, if we evaluate **d** at  $u = 1$ , we get 22, at  $u = -3.5$ , we get  $-0.5$ . Often we are interested in how the value of the expression changes as we change the value of  $x$  and evaluate, and so we generate a table of corresponding values of  $x$  and the expression, and go on to graph those points on a coordinate plane. Were we to do so, it always turns out that the points lie on a straight line, and that is why expressions of this form are called “linear”. But we are getting ahead of ourselves; we will return to this discussion in the next chapter.

It is often the case that different linear expressions have the same meaning: for example  $x + x$  and  $2x$  have the same meaning, as do  $x - x$  and 0. By the *same meaning* we mean that a substitution of any number for the unknown  $x$  in each expression produces the same numerical result.

#### EXAMPLE 2.

Consider these linear expressions:

$$\mathbf{a.} \ 2(x + 5) \quad \mathbf{b.} \ 2x + 10 \quad \mathbf{c.} \ x + 10 \quad \mathbf{d.} \ 2x + 5$$

**a.** and **b.** have the same meaning, since every substitution of a number for  $x$  gives the same result. As for **a.** and **c.**, although the substitution of 0 for  $x$  produces the same result, it will not work for any other number. And the situation for **a.** and **d.** is even worse: there is no number for which they both give the same result,

How do we know that **a.** and **b.** have the same meaning, since we cannot test *every* number? The answer lies in the laws of arithmetic. We say that two linear expressions are *equivalent* if we can move from one expression to the other using the laws of arithmetic. When two linear expressions are equivalent, they have the same meaning: to be precise, any number substituted for the unknown in the expression always returns the same value. Indeed, this is why the “laws of arithmetic” are called laws: they preserve the meaning of the expression. This is very convenient: to show that two expressions have the same meaning, we do not have to check every number (an impossible task in any event); it is enough to show that we can move from one expression to the other using the laws of arithmetic. On the other hand, to show that two expressions are not equivalent, just find one number that gives different results when substituted for the unknown in the expressions.

In this chapter we want to work precisely with these ideas of equivalence of linear expression, leading to *simplifying* and *solving*. In the next chapter we will see that the graph of a linear expression (where  $y$  is the value of the expression for a value of  $x$ ) is a straight line. Since a line is determined by any two points on the line, we will see that two linear expressions are equivalent, if they return the same values for any numbers substituting for the unknown.

Every linear expression is equivalent to one in the form  $ax + b$  by applying the laws of arithmetic. This process is sometimes called *simplifying* the expression. However, “simplify” is not always what we want to do with an expression. For example, if we change  $7x - 28$  to  $7(x - 4)$ , then we learn that in the equation  $y = 7x - 28$ ,  $y$  is

proportional to  $x - 4$ . So, if we are interested in the behavior of  $x$  and  $y$  in the equation  $y = 7x - 28$ , the form  $y = 7(x - 4)$  is “simpler.” In general, the word “simplify” should be tied to the goal of the work being done; right here it is to minimize the number of symbols necessary to understand the expression.

Every one of the above expressions in example 1 can be put in the form  $ax + b$  for some numbers  $a$  and  $b$ , using arithmetic operations. As for examples **a**, **b**, **c** and **f**, they are already in that form. Let’s now look at the others, and a few more just to review all the possibilities.

#### EXAMPLE 3.

- a. example 1d:  $3u + 2u + 17$  is equivalent to  $5u + 17$ . For we can combine similar terms: “three  $u$ ’s plus two  $u$ ’s” is the same as “ $5u$ .”
- b. example 1e: By doing the division implied by the fraction, we see that  $3x - \frac{12x-16}{4}$  is equivalent to  $3x - 3x + 4$ , which is just 4 (as mentioned above).
- c. example 1g: Distribute the 6 in  $6(2x - 5) + 11$  to get  $12x - 30 + 11$ , and now add  $-30 + 11$ , to get  $12x - 19$ .
- d.  $4x + 5$  and  $2x + 3x + 5$  are not equivalent. Substitute 1 for  $x$ , and obtain 9 in the first expression and 10 in the second.
- e.  $3x + 5$  and  $6x - 1$  are not equivalent: if we substitute 1 for  $x$ , we get 8 in the first expression and 5 in the second. But be careful: if we substitute 2 for  $x$ , we get the same result:  $3(2) + 5 = 11$  and  $6(2) - 1 = 11$ . Since we can find at least one value for the unknown that gives different results to the expression, they are not equivalent.

#### EXAMPLE 4.

$6x - 20 + 2(x - 4)$  and  $4(2x - 7)$  are equivalent.

Let’s go through the steps, giving clear reference to the relevant laws of arithmetic.

Step 1. Start with  $6x - 20 + 2(x - 4)$ . Distribute the 2 to remove the parentheses, to get:  $6x - 20 + 2x - 8$ .

Step 2. Combine like terms to get:  $8x - 28$

Step 3. Factor out 4 to get  $4(2x - 7)$  which is precisely the second expression.

There are many ways to go from one expression to an equivalent one. For example, we could distribute and collect terms in both expressions to obtain  $8x - 28$  from each. To put this another way: two expressions are equivalent if they are both equivalent to a third expression.

In summary, the end result of simplification is an expression of the form  $ax + b$ : This is always the case: any linear expression simplifies to the form  $ax + b$ . The word “simplify” is often ambiguous - it usually depends upon where it is you want to go with the expression; in this case, it is to the form  $ax + b$ .

## Section 1.1. Solving linear equations: obtaining the desired value of an expression

*Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms. 8.EE.7.ab*

In the introduction to this chapter, we talked about “evaluating expressions”. Here we ask: given a linear expression, and a number  $c$ , for what value of the unknown does the expression compute to  $c$ ? This can be restated as:

given the expression  $ax + b$  and a number  $c$ , find the value of  $x$  that produces that  $c$ . Let's first review what was done in grade 7.

**EXAMPLE 5.**

- a. For what  $x$  does  $2x + 5$  evaluate to 17? Otherwise put: solve  $2x + 5 = 17$ .

**SOLUTION.** Subtract 5 from both sides of the equation to get  $2x = 17 - 5$ . Replace  $17 - 5$  by 12 to get  $2x = 12$ . Now divide both sides by 2 to get  $x = 6$ .

- b. For what  $x$  does  $2(x + 5)$  evaluate to 24? Otherwise put: solve  $2(x + 5) = 24$ .

**SOLUTION.** Divide both sides by 2 to get  $x + 5 = 12$ . Now add 5 to both sides:  $x = 7$ .

Note that in the second problem, we'd rather not use the distributive property: it is easier and quicker to first divide by 2 than to distribute the 2.

Now we want to work more complicated expressions. The procedure will be the same, except that first we have to appropriately simplify the expression. Let's work with the expressions **e**, **f** and **g** of example 1.

**EXAMPLE 6.**

- a. For what value or  $x$  is  $3x - \frac{12x - 16}{4}$  equal to 5?

**SOLUTION.** First, we reduce the fraction to obtain the equation  $3x - (3x - 4) = 5$ , and then use the distributive property to get  $3x - 3x + 4 = 5$ . Then combine terms to obtain  $4 = 5$ . Since 4 is not equal to 5, there is no value of  $x$  to obtain 5 from this expression.

- b. Let's slightly change the expression so as to obtain a more satisfying result. For what value or  $x$  is  $5x - \frac{12x - 16}{4}$  equal to 25?

**SOLUTION.** Again, we reduce the fraction, this time obtaining  $5x - 3x + 4 = 25$ . Combining terms, this becomes  $2x + 4 = 25$ , which has the solution  $x = \frac{21}{2}$ .

**EXAMPLE 7.**

For what value of  $x$  is  $\frac{2}{5}x + \frac{3}{10}$  equal to 0.375?

**SOLUTION.** Otherwise put, solve

$$\frac{2}{5}x + \frac{3}{10} = 0.375$$

First, Multiply both sides by 10 to obtain

$$4x + 3 = 3.75$$

Subtract 3 from both sides getting  $4x = 0.75$ , and divide by 4 getting  $x = 0.1875$

This is a good time to point out that there can be many ways to solve a problem, and in this case, there may be better ways. Noticing that the notation is hybrid (we have both fractions and decimals) we could move to one notation or the other.

Yet another way would be to write all numbers as decimals to get  $0.4x + 0.3 = 0.375$  and now multiply by 10 to get  $4x + 3 = 3.75$ , and now proceed as above.

Write all numbers as fractions to get

$$\frac{2}{5}x + \frac{3}{10} = \frac{3}{8}$$

Multiply by 40 to eliminate denominators, getting

$$16x + 12 = 15$$

Subtract 12 from both sides to get  $x = \frac{3}{16}$

EXAMPLE 8.

a. For what value of  $x$  is  $6(2x - 5) + 11 = 53$ ?

b. For what value of  $x$  is  $6(2x - 5) + 11 = y$ ?

SOLUTION. a. We want to illustrate two different ways to solve this problem.

a1. Distribute the 6 and add  $-30$  to 11:  $12x - 19 = 53$ ,

Add 19 to both sides:  $12x = 72$ , and now divide by 12 to get  $x = 6$

a2. Subtract 11 from both sides :  $6(2x - 5) = 42$ ,

Divide by 6:  $2x - 5 = 7$

Add 5 to both sides:  $2x = 12$ ,

and now divide by 2 to get the answer  $x = 6$ .

b. First, apply the first two steps of a1 to get  $12x = y + 19$ . Now divide by 12 to get the result

$$x = \frac{y + 19}{12}$$

## Section 1.2. Solving linear equations: equating two expressions

A *linear equation* is an assertion that two linear expressions are equal. In the above, we have considered the case where one of the expressions is simply a number, and put it in the context of evaluation of expressions. Now we want to find out for what value of the unknown two expressions produce the same result. This may seem more difficult to the students, but the ideas are precisely the same. The difficulty may be this: it is clear that we can subtract 5 from both sides of the equation to get an equivalent equation, but since we don't know what  $x$  is, is it really all right to subtract  $5x$  from both sides? Of course it is, since  $x$  does represent a specific number, and so the laws of arithmetic apply. Later, when we move from the concept of "unknown" to that of "variable", then  $x$  is a quite different object, representing not some particular number that we don't know just yet, but any possible number. Nevertheless, the same reasoning applies: the laws of arithmetic actually do hold for any numbers and any expressions.

If a linear equation is an assertion that two linear expressions are equal, then “solving” the equation is to find out for what numbers the assertion is true. Two linear equations are equivalent if one can be obtained from the other by a succession of applications of laws of arithmetic. The goal of solving the equation is to find a sequence of equivalent equations, starting with the given equation and ending up with something like  $x = 5$ . Of course, we may not end up there: just as the expression  $x - x + 1$  is equivalent to the expression 1, the equation  $x - x + 1 = 2$  is equivalent to  $1 = 2$ , which of course is false. Since it is false, no matter what value  $x$  takes there is no solution.

In general, the result of this process may be “all numbers” or “a particular number” or “no numbers”. Let’s look at some examples:

EXAMPLE 9.

a.  $2(x - 5) = 3x - 1$     b.  $2(x + 5) = 2x + 10$     c.  $7 = 5$     d.  $7x = 5x$   
e.  $3(x - 5) = 2x$     f.  $7 = 5 + 2$     g.  $7x = 7x + 1$

The examples presented here are designed to indicate the breadth of issues that may come up as students learn this subject, and not to provide instructions. As you look through them keep in mind that the truth or falsity of the equation is something to be determined: it is our task. This is different from the validity of the equation as a statement. To illustrate: “Julius Caesar was the first President of the United States” is a valid statement, but false. The assertion “Mr. XXX was the first President of the United States” is a valid statement, but doesn’t tell us much (except that that person was male). The equally valid and true statement is “George Washington was the first President of the United States’.” Analogously, in example 8, **a.** is true for one value of  $x$ , **b.** for all values of  $x$  and **c.** for no values of  $x$ . If the equation is true for the substitution of every number for the unknown, it is an equivalence. Now, the reader might conclude that we cannot substitute *every* number for the unknown, so we can never be sure it is an equivalence. However, for linear equations, after we verify in the next chapter that the graph is a straight line, it follows from the fact that a straight line is determined by just two points, that we need only check two values of  $x$ . Now, if an equation is not an equivalence, it still may be true for some substitutions of  $x$  (these are called the solutions), or there may be no substitution to make it a true statement.

There are various techniques for solving a linear equation; all techniques amount to applying arithmetic operations to the equation that do not change the set of solutions. There are three kinds of operations:

**1. Apply the laws of algebra to simplify the expressions;** in particular, distribute to remove parentheses and combine like terms.

Transform the equation  $2x + 3x = 5 + 20$  to the equation  $5x = 25$ . Transform the equation  $6(x - 2) = 11$  to the equation  $6x - 12 = 11$ .

**2. Add or subtract the same expression to both sides of the equation.**

Transform  $3x = 2 - x$  to  $4x = 2$  by adding  $x$  to both sides.

**3. Multiply or divide both sides of an equation by a nonzero number.**

Transform the equation  $2x = 8$  to  $x = 4$  by dividing both sides of the equation by 2. Transform  $3x = 6x - 18$  by first dividing by 3 to get  $x = 2x - 6$ , and then combine like terms to find  $x = 6$ .

These operations all transform any equation into another with the same set of solutions. What is most important is that they are effective: they succeed in solving any linear equation. Let’s apply these ideas to the equations of parts **a** through **g** of example 8.

EXAMPLE 8 SOLUTIONS.

a.  $2(x + 5) = 3x - 1$ ;

Simplify the left side:  $2x + 10 = 3x - 1$ ;

Subtract  $2x$  from both sides:  $10 = x - 1$ ;

Add 1 to both sides:  $11 = x$ . Thus there is one solution:  $x = 11$ .

b.  $2(x + 5) = 2x + 10$ ;

Simplify the left side:  $2x + 10 = 2x + 10$ . Since both sides are the same expression this is true for all values of  $x$ ; that is, the expression on both sides of the equals sign in b) are equivalent. Consequently, every number is a solution to this equation.

c.  $7 = 5$  this is false: If we think of this as  $7 + 0x = 5 + 0x$ , we can assert that there is no value of  $x$  to make it true.

d.  $7x = 5x$ : Subtract  $5x$  from both sides:  $2x = 0$ . Divide both sides by 2:  $x = 0$ , so 0 is the only solution.

e.  $3(x - 5) = 2x$ ;

Simplify the left side:  $3x - 10 = 2x$ ;

Subtract  $2x$  from both sides:  $x - 10 = 0$ ;

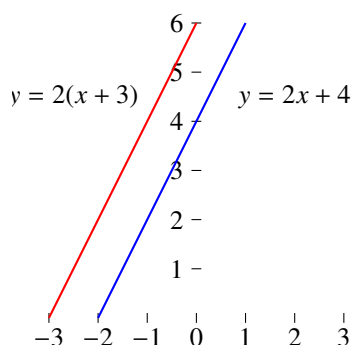
Add 10 to both sides:  $x = 10$ . Therefore, there is one solution:  $x = 10$ .

f.  $7 = 5 + 2$ : combine like terms on the right:  $7 = 7$ . This is a true statement.

g.  $7x = 7x + 1$ : Subtract  $7x$  from both sides:  $0 = 1$ . This is a false statement, so there is no solution to the original statement: no choice of value for  $x$  will make it true..

Let us take a moment to notice the exceptional cases: **b**, **c** and **g**, where we do not get a solution, but either get an equivalence (all numbers solve the equation) or there is no solution. These occur when the coefficient of  $x$  is the same on both sides of the equation. We see this in **b** after distributing the 2, and in **c** and **g** the equation starts that way.

There is a geometric representation of this situation that helps make this clear. Suppose that we start with the expressions  $2x+4$  and  $2(x+3)$ . Let's graph the two expressions: that is, graph the equations  $y = 2x+4$  and  $y = 2(x+3)$ .



In general, when we graph two linear expressions, we get two lines, and the point of intersection gives the value of  $x$  for which the two expressions give the same number. However in this case (see the figure), the lines are parallel, so there is no point of intersection; explaining why there is no solution. Now, if we had started out with the expression  $2x + 4 = 2(x + 2)$ , then the figure would have shown just one line, since the two expressions are equivalent.

An important feature of the allowable operations on equations is that they can be reversed: if an operation takes one equation to another, it can be undone, meaning that there is an operation on the second equation that produces the first.

EXAMPLE 10.

Solve  $-3x + 8 = 20 + x$ .

Step 1: Subtract  $x$  from both sides to get:  $-4x + 8 = 20$ .

Step 2: Divide both sides by  $-4$  to get  $x - 2 = -5$ .

Step 3: Add 2 from both sides to get  $x = -3$ .

Now let's reverse the process. Start with  $x = -3$ .

Step 1: Subtract 2 from both sides to get  $x - 2 = -5$ .

Step 2: Multiply both sides by  $-4$  to get  $-4x + 8 = -20$ .

Step 3: Add  $x$  to both sides to get  $-3x + 8 = -20 + x$ .

**Summary:** To solve any linear equation, use these rules, not necessarily in the order listed. Practice develops a sense of the sequence that best leads to the solution:

1. Use the distributive law to remove parentheses.
2. Combine like terms so that each side of the equation is of the form  $ax + b$ .
3. Add the same expression to both sides of the equation so that  $x$  appears on only one side of the equation.
4. Divide by the nonzero coefficient of  $x$ ; resulting in an equation of the form  $x = c$ .

EXAMPLE 11.

Students will need to develop a facility for discovering mistakes in the procedure, when, after checking, it is discovered that the arrived at answer does not solve the original equation. In the following determine whether or not the following arguments are correct, and if incorrect, explain the error,

**a.**  $2(x + 5) = 13$

$$2x + 5 = 13$$

$$2x = 8$$

$$x = 4$$

**b.**  $3x - 15 = 24$

$$3(x - 5) = 24$$

$$x - 5 = \frac{24}{3}$$

$$x - 5 = 8$$

$$x = 13$$

**c.**  $2x + 3 = x + 10$

$$2\left(x + \frac{3}{2}\right) = x + 10$$

$$x + \frac{3}{2} = \frac{x}{2} + 10$$

$$\frac{x}{2} = 10 - \frac{3}{2}$$

$$x = 10 - 3 = 7$$



## SOLUTION.

a. There is a mistake in the first step: the 2 is improperly distributed; the second line should be  $2x + 10 = 13$ . Now subtracting 10 from both sides gives  $2x = 3$  which leads to the answer  $x = 3$ . Now check:

$$2\left(\frac{3}{2} + 5\right) = 2 \times \left(\frac{3}{2}\right) + 2 \times 5 = 3 + 10 = 13$$

b. This is not so straightforward as the preceding problem, so first we check whether or not the answer satisfies the first equation: substitute 13 for  $x$  to get  $3(13) - 15$  which is  $39 - 15$ , which simplifies to 24. Since this is the same as the right hand side of the equation, the answer is correct. But that does not mean that the argument is correct: we have to still check that the step from one line to the next is a correct application of arithmetic. In this case, it checks out: line one to two is the distributive property; line two to three: both sides are divided by 3; line three to four, replaces  $24/3$  by the equivalent number 8 and finally we get to the last line by adding 5 to both sides.

c. First, let's check that 7 solves the first equation: with this substitution for  $x$ , the left hand side is  $2(7) + 3 = 17$ , and the right hand side is  $7 + 10 = 17$ . Now, let's check that the reasoning is correct. Well, there is a mistake in going from line two to three: we are dividing both sides by 2, but on the right, the second term (10) was not divided by two. The next step is correct: we have subtracted  $x/2$  and  $3/2$  from both sides. However, the step to the last line is faulty: in multiplying both sides by 2, we failed to multiply the term 10 by 2. This error has the effect of correcting the preceding error, so when we end up with  $x = 7$ , we accidentally ended up with the correct answer even though the steps were flawed.

## Section 1.3. Creating and Solving Linear Equations to Model Real World Problems

*Solve real world problems with one variable linear equations. 8.EE.7c*

Up to this point in this chapter we have been discussing the questions: what does an equation tell us, "What does an equation tell us?", "What does it mean to solve an equation?", and "What is the process for solving linear equations?" However our students are most unlikely to ever see an equation to solve except in other math classes that they will take or, possibly, will teach. So why are we teaching this material? The answer is this: they will have, as professionals, workers, and as human beings living in the 21st Century, many problems to solve on a daily basis. They will need tools to think through those problems in a way that leads to an acceptable course of action. If the course of action depends upon the values of specific quantities, it is highly likely that algebra - or more often, algebraic thinking - will be the tool to apply.

It is coherent and effective algebraic thinking that is the goal of the study that started in sixth grade and continues through to high school graduation. It is *not* the protocol, or technique: we have computers and calculators to do that work. So then, the question again arises: why do we teach the protocol for solving equations? One might just as well ask: "Why does the plumber have to know his wrenches?", "Why does the dentist have to understand the dentist chair?", "Why does a carpenter have to know what a hammer can and cannot do?", "Why do we have to learn about our car before we can drive it?" The answers are so obvious that these questions hardly ever come up. But the connection between algebraic thinking and solving of real-life problems is not easily taken for granted, and one must be able to explain it - the teacher in order to scaffold the teaching, and the student in order to be motivated to learn.

Although this is the third section on linear equations, it is the most important one: the content here is the relationship between a problem stated verbally, and its algebraic restatement. This we call *mathematical modeling* of a problem. The state of one's financial resources may suggest that, while shopping, one take into consideration the relationship between cost and value. This may involve gathering of data (more shopping) and then analysis of those data. All of these operations involve algebraic thinking.

Modeling of a problem makes it possible to do calculations and make predictions. For example, we record the passage of time with clocks and watches, and make calculations based on those observations, conceptually using the model of a needle moving continuously along the real line in the positive direction. But we should not take the model too seriously: if we are told that the new record for running a mile is 3 minutes and 37.65 seconds, we should keep in mind that that is an approximation, as will be any response, no matter how accurate the timepiece. If in the next race, the same runner runs the mile in 3 minutes and 37.65 seconds according to the chronometer available, we say that the time is “the same,” but we can neither claim that this is precise, nor that one race was run faster than the other - unless some other observer had a more precise chronometer.

Students should understand the power they have when they are able to move fluently between the verbal description and symbolic representation of a linear context. What does the symbolic representation allow them to do? Symbolic representations allow students to show the relationship between the quantities, solve problems, draw conclusions, make decisions, etc.. Of what must students be careful when they translate to a symbolic representation? Are they clear in what they have defined their variable to represent in the context? What quantities and units are involved? Once solved, can they interpret the solution in the context? Often, when students formulate the algebraic representation and solve it, they struggle to interpret that answer in its context and for confidence that they have even answered the question(s) being asked.

Being able to figure out how to attack problems as they come up, and what protocols to use to solve them, is the central goal of education. Students need to know how the protocols work in order to be able to formulate the problem so that a protocol can be applied. In the following set of problems we concentrate on how to get from a verbal problem to the equation that represents it.

#### EXAMPLE 12.

Here is a game, based on the preceding sections, that might attract the students’ attention to the uses of algebra. Pick a number from 1 to 20. Add three and double the result. Add 8. Take half of that number. Subtract your original number. You have a 7, right?

Let us analyze this “trick.” The subject picks a number; since it is a number but we do not know it, we call it  $N$ . Then the subject is asked to perform a set of operations, leading to the following list:

<i>Operation</i>	<i>Result</i>
Add 3 :	$N + 3$
Double :	$2(N + 3)$
Add 8 :	$2(N + 3) + 8$
Take half :	$N + 3 + 4$
Subtract original number :	$3 + 4$

So when you, the “magician,” reveal that the number in hand is 7, we see that it is algebra, not magic. If this trick is played several times, it is a good idea to change the number 8 to another even number  $2K$ . Then the end result is “revealed” to be  $3 + K$ .

A variant of this that may seem a little more magical is this: at the  $N + 3 + 4$  stage, ask for the number the subject now has. When this is received, mentally subtract 7 and says: “Your original number was . . . .”

#### EXAMPLE 13.

- a. Fred, Jon and Brody pooled their resources for a \$150 room at a resort. Jon put in twice as much as Fred, and Brody put in \$10 less than Fred. How much did each put in?

In this problem we are asked for the amount each put in the pool. In order to pick the “unknown,” it may be necessary to read the problem more carefully. The statements all relate the unknowns to Fred’s contribution, so we should select that as the primary unknown; let’s call it  $F$ . Now James put in twice

as much as Fred, so he put in twice  $F$ ; that is, his contribution is  $2F$ . Brody put in \$10 less than Fred, so his contribution is  $F - 10$ . The sum of these numbers is \$150, and this becomes the equation:

$$F + 2F + (F - 10) = 150 .$$

We now assign this to our assistant, an expert on the preceding two sections of this chapter, who comes up with the answer :  $F = 40$ , so Fred put in \$40, Jon put in \$80, and Brody put in \$30.

Here is a slightly more complicated variant:

- b.** Fred, Jon and Brody pooled their resources for a \$150 room at a resort. Jon put in twice as much as Fred, and Brody put in \$10 less than Jon. How much did each put in?

We are still looking for three numbers, but it is no longer true that the other two are directly related to Fred's contribution. But Brody's is related to Jon's, and Jon's is related to Fred, so it still all comes down to Fred. Again, if we denote Fred's contribution as  $F$ , then that of Jon is  $2F$ . Now Brody put in \$10 less than Jon's, which is  $2F$ , so Brody's contribution is  $2F - 10$ . Now add them up to get \$150:  $F + 2F + 2F - 10 = 150$ , leading to the result  $F = 32$ .

This problem illustrates several possible sources of confusion, centering around the interpretations of statements like "A is four less than B" and "A is twice B." As for the first, it is a problem of the language: there is a significant difference between the statements "A less four is B," and "A is four less than B" The students may have trouble with the linguistic difference; this can be resolved by testing with real numbers: take  $A = 5$  and  $B = 1$ . Now, which is true: "5 less 4 is equal to 1" or is "5 is 4 less than 1"? The equation expressing the first statement is " $5-4=1$ ," which is true, and for the second we get " $5=1-4$ ," which is false. In the same way, we test the second statement: A is twice B. Take  $A = 4$  and  $B = 8$ , put them in the statement, and pick the version that is true.

#### EXAMPLE 14.

A salesman at the XYZ car dealership receives a salary of \$1,000 per month and an additional \$250 for each car sold. How many cars should he sell each month so as to earn \$8,000 in a month?

**SOLUTION.** The way to the solution is to find out the relationship between income and number of cars sold. Test some numbers: If no cars are sold, the salary is just the base \$1000; if one care is sold, the earnings are \$1250. Look at some more test cases to discover the pattern:

- a.** if this salesman sells 4 cars, his income for that month is:  $1000 + 4(250)$ ;
- b.** if this salesman sells 12 cars, his income for that month is:  $1000 + 12(250)$ ;
- c.** if this salesman sells  $N$  cars, his income for that month is:  $1000 + N(250)$ .

The last expression simply amounts to recognizing that the computation for 4 or 12 or "no matter how many" cars sold is the same.

In this problem we want the income to be \$8,000, so we look at **c.**, set it equal to 8000 and hand it over to our algebra assistant for the solution.

#### EXAMPLE 15.

Lucinda is on the school track team; she can run 8 miles in an hour. Her younger sister Josefa isn't yet a runner, but can walk at a pace of 3 miles an hour. Josefa left their home forty minutes ago heading

toward downtown, and Lucinda wants to catch up with her. If she runs how long will it take to catch Josefa? How far will they be from home?

**SOLUTION.** First, we focus on the first unknown that we have to determine: in this problem it is the time Lucinda needs to catch Josefa; let's call that  $T$ , in hours. In that time Josefa has walked  $3T$  miles and Lucinda will have run  $8T$  miles. So, after  $T$  hours, Lucinda is  $8T$  miles from home; but since Josefa was already 2 miles down the road when Lucinda started, Josefa is  $3T + 2$  miles from home. When they actually meet these two distances have to be the same, giving us the equation

$$8T = 3T + 2$$

The solution of this equation is  $T = (2/5)$ . Now, it is important to recall what this means: what was  $T$  and what are the units for  $2/5$ ? As soon as the algebra takes over, meaning of the symbols becomes irrelevant, but when we've solved the algebraic equation we must return to the meaning of the symbol to fully understand what we have discovered:  $T$  is  $2/5$  of an hour, or 24 minutes. It takes energy, once the "math" is done, to return to focus to the problem. But when we do, we see there is another part of the problem: How far away have they gone. Well, Lucinda ran for  $2/5$  of an hour and she runs at 8 miles per hour, so she ran  $(2/5)8$  miles, which is five and a third miles.

#### EXAMPLE 16.

My new hybrid car can get 35 miles to the gallon. Gas costs \$3.25 per gallon. San Francisco is 825 miles from here. How much will I spend on gas to drive to San Francisco?

**SOLUTION.** Let  $C$  be the cost of driving there. We have to relate  $C$  to miles, denoted by  $M$ , and the information we are given is how  $C$  relates to gallons, denoted by  $G$ , of gas:  $C = 3.25G$  and how miles relate to gas:  $M = 35G$ . This tells us that  $G = M/35$ , and so we can substitute that in the first equation to get  $C = 3.25(M/35)$ , or  $C = 0.093M$ . Since we want to go 825 miles, the cost of gasoline will be  $C = 3.25(825) = 0.093(825) = 76.725$ , or \$76.73.

The preceding example is, in part, an example of a *literal* problem: one in which several quantities are related and we want to express that relationship by a formula. So, in that example, we want to express the cost of gasoline ( $C$ ) in terms of miles ( $M$ ), already knowing the cost of gasoline per gallon and the number of miles per gallon; and we ended up with the relation  $C = 0.093M$ , or 9.3 cents per mile.

Here is another literal problem:

#### EXAMPLE 17.

Let's return to example 14, of the salesman at the XYZ dealership. We saw (see part c)) that if the salesman sells  $N$  cars in a month, then his compensation is  $1000 + 250N$ . The salesman may ask: how many cars do I have to sell to have an income of  $C$  in a given month?

**SOLUTION.** The relationship between compensation ( $C$ ) and number of cars sold ( $N$ ) is  $C = 1000 + 250N$ . The salesman wants to know what  $N$  should be to attain a certain value  $C$ , so he wants a formula that calculates  $N$ , given a value of  $C$ . These are the steps:

$C = 1000 + 250N$  : this is the starting relationship.

$C - 1000 = 250N$  , subtract 1000 from both sides.

$\frac{C - 1000}{250} = N$ , divide both sides by 250.

This is the formula to calculate the number of cars to be sold to earn  $C$  dollars. So, for example, if the salesman wants to earn \$25,000 in a particular month, he must sell  $N = (25000 - 10000)/250 = 96$  cars.

**To summarize: here is a procedure for solving problems:**

1. Read the problem carefully, making sure to identify the unknown(s).
2. Recognize the information in the problem that can be translated into mathematical expressions or equations.
3. Apply the rules for solving linear equations.

**EXAMPLE 18.**

The conditions of John's job are that he can work whenever he wants to, but in any day he works, he receives no compensation for the first three hours of work, and \$20 per hour for each subsequent hour. On one particular day he wants to buy a special shirt for \$190. He has \$70 in his pocket. How many hours must he work on that day to be able to buy the shirt?

**SOLUTION.**

1. What we want to find is the number of hours to work so that the income, together with the \$70 he starts with comes to \$190. Let  $N$  be the unknown: the number of hours he has to work.
2. For  $N$  hours of work, he receives no income for the first three hours, and \$20 for each subsequent hour. Thus he receives \$20 for each of  $N - 3$  hours.
3. After  $N$  hours of work he has earned  $20(N - 3)$ . That added to the \$70 he started with is to provide the \$190 needed to buy the shirt. Thus  $N$  must satisfy

$$20(N - 3) + 70 = 190.$$

Our assistant plugs and chugs and finds that  $N = 9$ . John must work 9 hours in order to have enough money to buy the special shirt.

# Chapter 2

## Exploring Linear Relations

In the preceding chapter we completed the topic of finding solutions of a linear equation in one unknown. In chapter four we will turn to this study of techniques to find solutions for a pair of linear equations in two unknowns. But now we want to turn to another thread started in previous grades, that of representing and understanding linear relations in two variables. Notice the change in language: from *equation* and *unknown* to *relation* and *variable*. This is a significant change in objective: from that of finding specific numbers that satisfy given conditions, to that of understanding how conditions on the relation of two variables determine how they behave with respect to one another. In seventh grade students studied the properties of a proportional relation between two variables; in this chapter we turn to linear relations between two variables. A significant tool is the graphical representation of a linear relation by a straight line, leading to the correspondence between *rate of change* (for the relation) and *slope* (of the line).

There are two ways to bring together the study of proportional relations and the solution of linear equations in order to understand linear relations, one emphasizing geometric aspects and the other emphasizing the algebra. Algebraically, linear relations are generalizations of proportional representations: we replace the equation  $y = mx$  by the equation  $y = mx + b$ . The commonality between these is that the *rate of change* of  $y$  with respect to  $x$  is a constant; the difference is that for a proportional relation, the quotient  $y/x$  is constant, and is called the *unit rate* of  $y$  with respect to  $x$ . For a linear relation, the quotient  $y/x$  is not constant unless  $b = 0$ . Here,  $b$  is considered the *initial value* of  $y$ ; that is, the value of  $y$  corresponding to  $x = 0$ . Geometrically, linear relations and proportional relations are both represented by straight lines; the difference is that the graph of a proportional relation goes through the origin, while the graph of a linear relation goes through the point  $(0, b)$ , called the *y-intercept*. So, a proportional relation is a special case of a linear relation. In particular, if we slide the graph of  $y = mx + b$  by the amount  $b$ , we get a line through the origin, and thus the graph of a proportional relation. This just realizes the fact that in the linear relation  $y = mx + b$ , the quantities  $y - b$  and  $x$  are proportional, with  $m$  the constant of proportionality.

The facts that there are these two ways of developing the subject of linear relations, that both approaches are important, and that the differences are subtle, create a learning issue: the student has to assimilate the two approaches at the same time and appreciate the subtle differences between them. Our solution is to present both approaches, the geometric in the workbook and the algebraic in the foundation. This directly exposes the two approaches and gives the teacher the freedom, and obligation, to develop them simultaneously in a way that works best in that classroom.

Here we begin by continuing the study of proportional relations from seventh grade, focusing on the *unit rate* as a rate of change of one quantity with respect to the other. There will be a shift in language as we move from calculating values of quantities in a proportional relation, to the study of the relation itself. For example, we now consider the unit rate as the *constant of proportionality* of the relationship in order to emphasize that it is what remains *constant* while the measure of the quantities vary, and therefore are called *variables*. We observe that the graph of  $y$  vs.  $x$ , when  $y$  and  $x$  are in a proportional relation, is a straight line through the origin.

Then, we return to the study of linear expressions, but this time in the form of the function  $y = mx + b$  (although

we do not introduce the word “function” until chapter 3). We observe that the graph of  $y$  vs.  $x$  is a straight line that crosses the  $y$ -axis at the point  $(0, b)$ . By examination of tables of values and the graph, we observe that, although the variables are not proportional, their changes from one measurement to another are proportional; that is, the quotient of the change in  $y$  values with respect to the  $x$  values is constant (independent of the points chosen for the computation). This is called the *rate of change*. The constancy of the rate of change along the graph is a defining property of a straight line, as we shall see in section 3. The move from proportional relations to linear relations, and the accompanying shift from *unit rate* to *rate of change* is subtle and may be difficult for students to appreciate at this time. For this reason, we feel that it is essential to develop the subject in contextual examples, moving the algebraic formulation in the next chapter.

Although we have observed that the graph of a linear relation is (to be precise, *appears to be*, since we can only plot a finite number of values) a straight line, we still need to understand why this is so. In addition, we need to understand why a straight line is the graph of a linear relation. For this, we seek an algebraic characterization of a straight line, and to get there we have to begin with geometric ideas. Here we introduce *dilations*: transformations of the figures in the plane that retain “shape” but not “size.” These properties will be examined in detail in chapter 9; for the present purpose it suffices to observe that a dilation takes a right triangle with horizontal and vertical legs to another such triangle, and that the lengths of the corresponding sides of the triangles are proportional.

If we draw a line in the plane (that is neither horizontal, nor vertical) and then pick two points on the line, the segment of the line between those two points is the hypotenuse of a right triangle with vertical and horizontal legs. This we call *the slope triangle* for that segment. If we now draw the slope triangle for another pair of points, we can exhibit a dilation that takes one triangle to the other. The fundamental property of dilation is this: the length of line segments and the length of the images are proportional, with constant of proportionality the *factor* of the dilation. We conclude from this that the slope of any slope triangle on a given line is constant. In other words, the slope of a segment on the line is constant, and this is called the *slope* of a line. The logic of going from showing that any two computations give the same slope to the statement that slope is constant along the line is a bit subtle, and it might be a good idea to create other examples of that logic.

This leads directly to a way to calculate the *equation of a line*, which is the algebraic expression of the relation between the variables  $y$  and  $x$  graphically expressed by saying that they lie on a line. The outcome, which will be explored in detail in the next chapter is this: for a line  $L$  and two points  $P$  and  $Q$  on  $L$ , construct the slope triangle whose hypotenuse is the segment  $PQ$ . Let  $m$  be the slope of that segment. Now, let  $(0, b)$  be the point on the  $y$ -axis that lies on the line. Then the equation of the line is  $y = mx + b$ .

Figure 1 illustrates the geometry in this discussion. We show the cases for both positive and negative slope, to emphasize that slope is not the ratio of the lengths of the sides of the slope triangle, but the ratio of the *changes* in the variables. Thus, when the change in  $y$  is negative for the corresponding increase in  $x$ , then the slope will be negative.

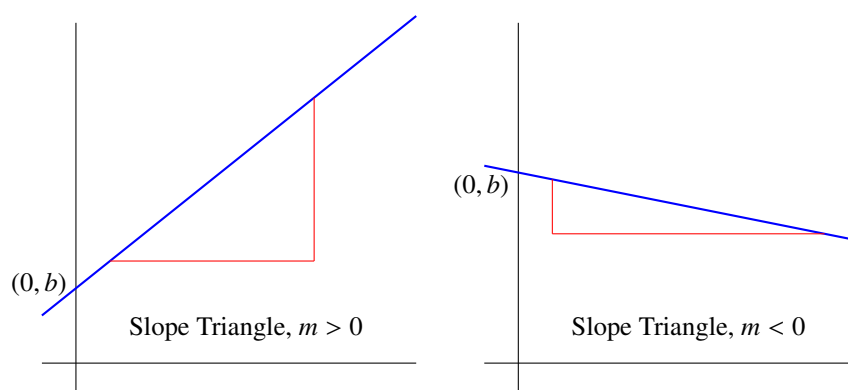


Figure 1

## 2.1 Linear Patterns and Contexts

### Proportional Relationships

*Graph proportional relations, interpreting unit rate as the slope of the graph, which is a straight line. 8.EE.5.1*

*Compare two proportional relationships represented in different ways (tables, graphs, equations). 8.EE.5.2*

The ideas of ratio and proportion were introduced in grade 6 and further developed in grade 7. In this section, after a brief review of the development of these ideas, we move on to the relation “proportional” (from the focus on the values of variables “in proportion.” This complements the gradual move away from the language of “unknowns” and “equations” to that of “variables” and “relations.”

In grade 6 the concept of *ratio* is introduced as a way of describing a relation between two collections of objects without reference to the actual size of those collections. So we may say that, in U.S. population, the ratio of minors to adults is 2:5, meaning that there are 5 adults for every two minors. This knowledge tells us, for example, that if we collect together a random group of people of size 140, we should expect 40 of them to be minors.

An old joke says that a shepherd keeps track of the herd by counting the legs and then dividing by 4. In terms of ratios, this expresses the fact that the ratio of sheep to sheep legs is 1:4. Actually, in Utah and Nevada where there are sheep herds numbering in the thousands, the shepherd keeps track of the herd by counting the black sheep and then multiplying by 40. This works for two reasons. First, sheep are social animals congregating in groups, so a count of sheep that estimates the actual number of the sheep within the size of a group has not missed any groups of sheep (maybe a stray lamb or two, but then the mother ewe will notify the shepherd of her distress). Second is that the ratio of sheep to black sheep is, for reasons of genetics, 40:1.

The concept of ratio (used mostly in counting individuals in particular sets) leads to the concept of *proportion*, which is more convenient than ratio for quantities in question can take on all numerical values, not just integral values.

- Given two quantities  $x$  and  $y$ , they are said to be *proportional* if, whenever we multiply one by a factor  $r$ , the other is multiplied by the same factor,  $r$ . For example, if we double the variable  $x$ , then  $y$  also doubles.

If two quantities are measuring the same physical attribute, they are going to be proportional. When we measure the length of a rod, we may do so in yards ( $Y$ ), or in feet ( $F$ ). Since these are measures of the same physical characteristic, they have to be proportional: if the rod triples in size, then its measure in feet or in yards also triples in size. A yard is defined as being 3 feet long, so we say that the ratio of yards to feet is 1:3. This can be rephrased as a proportional relationship with the *unit rate*: 3 feet per yard.

- If quantities  $y$  and  $x$  are in proportion then the *unit rate* of  $y$  with respect to  $x$  is the amount of  $y$  that corresponds to one unit of  $x$ . If  $m$  is the unit rate, then for any value of  $x$ , the corresponding  $y$  value is  $mx$ . If we interchange the roles of  $y$  and  $x$ , we would speak of the unit rate of  $y$  with respect to  $x$ . These two numbers are inverses of each other.

Since the unit rate of feet to yards is 3, the unit rate of yards to feet is  $1/3$ .

#### EXAMPLE 1.

There are 5280 feet in a mile. How many yards are in a mile?

SOLUTION. 1 mile = 5280 feet  $\times \frac{1 \text{ yard}}{3 \text{ feet}} = \frac{5280}{3}$  yards = 1760 yards.



In seventh grade, the unit rate is reinterpreted as the *constant of proportionality*. This corresponds to the change of focus from specific instances of a proportional relationship to that of the relationship itself. This leads to the equation  $y = mx$ , where  $y$  and  $x$  are the quantities in the proportional relation, and  $m$  is the constancy of proportionality. When two variables are proportional, all we need to know is one specific pair of values  $(x_0, y_0)$  in the relation to be able to compute all such pairs of values, for the ratio  $y_0/x_0$  gives us the value of  $m$ . Graphically this is clear: if we know a pair of values  $(x_0, y_0)$  in the relation, all pairs  $(x, y)$  in the relation lie on the line joining  $(0, 0)$  to  $(x_0, y_0)$ . So all we need to do is to draw the line joining the origin to the given point,  $(x_0, y_0)$ .

#### EXAMPLE 2.

The concepts of *ratio*, *constant of proportionality* and *unit rate* seem interchangeable, since they can all be represented by the same fraction, and this causes a lot of confusion with students. The way to address this confusion is to first understand that they are interchangeable, and are used in different ways in different contexts. So the second step in addressing this issue is to understand that the fundamental difference among these concepts is that they present different ways of looking at a problem in context, and that one has to learn how to decide which interpretation is relevant for a given context. Let us illustrate.

In basketball, it is necessary to have 12 players in a roster. In a particular district in Eastern Utah, the middle school basketball league has teams that are made up of boys and girls. For fairness, it is decided that each team must have 7 girls and 5 boys. This tells us that the ratio of girls to boys in the basketball league is 7:5. The relation, girls to boys in the basketball league is a proportional relationship, with constant of proportionality “girls to boys” equal to  $7/5$ .

Question 1. The district decides to have 8 teams in the league. How many girls and boys are there in the competition? This problem guides us to think in terms of the ratio 7:5: since there are 8 teams, each of which has 7 girls and 5 boys, the total number of players are  $8 \times 7 = 56$  girls, and  $8 \times 5 = 35$  boys.

Question 2. There are 45 boys eligible for basketball. How many girls are needed to complete the league? Here we want to think in terms of the constant of proportionality, which is  $7/5$ . So the number of girls needed is  $7/5$  of the number of boys available; that is,  $(7/5) \times 45 = 7 \times 9 = 63$ .

Here is a different problem: my grandfather drives at exactly 30 miles per hour.

Question 1. If Gramps drives 5 hours, how far does he go? Here, we think of unit rate: the rate of miles per hour is 30. Since

$$\text{miles} = \frac{\text{miles}}{\text{hours}} \times \text{hours} ,$$

he traveled  $30 \times 5 = 150$  miles. Question 2: Gramps wants to drive to St. George from Logan; that is 440 miles. How long will it take him at that rate. Here we want to convert to minutes, and the concept of ratio: the ration of minutes to miles is 2:1. So to drive 440 miles, takes Gramps 880 minutes, or  $880/60 = 14.6667$ , or 14 hours and 40 minutes.

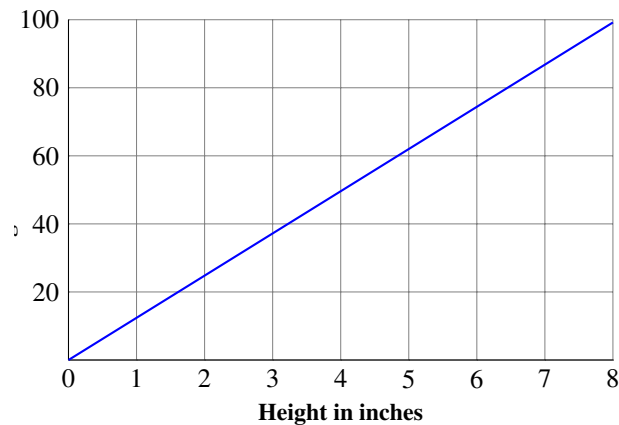
#### EXAMPLE 3.

To illustrate the development of a proportional relationship, consider measuring the amount of water in a cylindrical container (a glass or can). If we put a quantity of of water in the cylinder, we record the height of the column of water,  $H$ , and the weight  $W$  of the column of water. Both of these are measures of the amount of water: if we double the amount of water, both the height and the weight double.

Suppose that this experiment is done with a quantity of water, filling the cylinder a bit at a time, and each time, measuring both  $H$  and  $W$ . A table of the data would look something like this:

Height	0	2	3	4	6	8	inches
Weight of Container	2.5	2.5	2.5	2.5	2.5	2.5	ounces
Measured Weight	2.5	27.3	39.7	52.1	76.9	101.7	ounces
Weight of Water	0	24.8	37.2	49.6	74.4	99.2	ounces

Notice that, we have accounted for the weight of the container itself by first measuring it empty, and then subtracting that weight from the weight of the container and water at each measurement. We now graph the these data, plotting height along the horizontal axis and weight on the vertical: The graph appears to be a straight line, giving confirmation of our hypothesis that the height of the column of water and its weight are proportional. We can calculate the unit rate of change using any one of the measurements: for example, 4 inches of the water weighs 49.6 ounces, so we have 12.4 ounces per inch of water. This is expressed by the relation  $W = 12.4H$ . In an actual experiment there always will be slight variations due errors or estimation, given the accuracy of the instruments used. So, the rates computed from each measurement may differ slightly. We will return to this in the statistics chapter.



Graph of Example 2 Data

In summary, the height of the column of water in the container and its weight are two different ways of measuring the volume of the column of water. Since the volume of the water is the same no matter how it is measured, the measurements are related. Similarly, yards, feet, inches and meters are different ways of measuring lengths; ounces, pounds, grams are different ways of measuring weight. All these relations have the property that a doubling or halving of the object (volume of water or length of stick) has the effect of doubling or halving the measure. In fact if the amount of the object is changed by the factor  $a$ , then any measure of the object also changes by the factor  $a$ . When quantities are related in this way, we say that they are *proportional*.

#### EXAMPLE 4.

If we are told that  $x$  and  $y$  are in the relationship  $y = 7x$ , then  $(1,7)$ ,  $(2.5, 16.5)$ ,  $(8,56)$  are all in this relationship, because the ratio of the  $y$  value to the  $x$  value is always 7.

#### EXAMPLE 5.

There are 5280 feet in a mile, so Feet/Miles = 5280, or Feet = 5280×Miles. To find out how many feet are in a quarter mile, let  $f$  represent that number of feet. Then we have  $f = 5280(1/4) = 1320$  feet. In yards, that is  $1320/3 = 440$  yards.

### EXAMPLE 6.

We have made measurements of two quantities, and formed this table:

x	0	2	4	5	7	8	10
y	0	3.6	7.2	9	12.6	14.4	18

The graph of these data appears to be a straight line through the origin suggesting a proportional relationship: Notice that whenever the value of  $x$  doubles, so does the value of  $y$ , and that a change in  $x$  of 1 unit is accompanied by a change in  $y$  of 1.8 units. Finally, when we calculate the quotient  $y/x$  for any pair of points, we always get the value 1.8. This can be phrased this way: the proportional relationship  $y = 1.8x$  models the given data.

- If quantities  $y$  and  $x$  are in proportion then the graph of pairs  $(x, y)$  in this relation will be a straight line through the origin. That line is characterized by the assertion that  $y/x$  is constant, and in fact, is the constant of proportionality. In terms of the graph, we call this its *slope*.

### Linear relationships

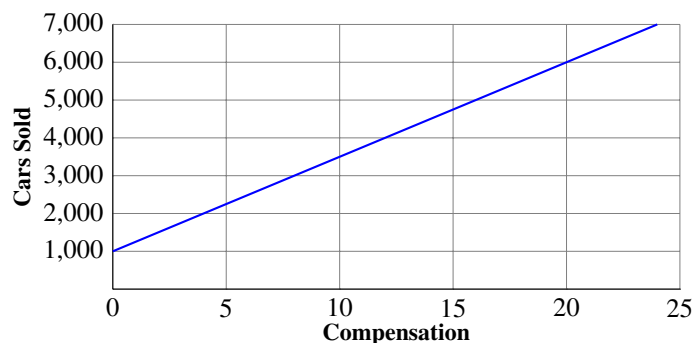
*Construct a function to model a linear relationship between two quantities. 8.F.4*

In chapter 1 we concentrated on solving linear equations of the form  $y = \text{linear expression}$ , where  $y$  is either a number or another linear expression. We also compared two linear expressions by graphing them (see the figure on page 6 of chapter 1)), and found that the graph of each linear expression is a line. In this chapter our goal is to see why there is this correspondence between linear expressions and lines. Let's start by taking another look at example 13 of Chapter 1.

### EXAMPLE 7.

A salesman at the XYZ car dealership receives a base salary of \$1000/month and an additional \$250 for each car sold. How many cars should he sell each month so as to earn a specified amount each month? There we ended up with this formula:  $C = 1000 + 250N$ , where  $N$  is the number of cars sold in a month, and  $C$  is the compensation received. Let's make a table for some possible values of  $N$  and then graph the result:

N	0	4	8	12	16	20
C	1000	2000	3000	4000	5000	6000



Cars Sold and Compensation

The graph (see the figure above) is a straight line that does not go through the origin: even if the salesman

sells no cars, he receives the base salary of \$1000. Also note that to each increment of 4 cars sold, the salesman receives an increase of \$1000. In particular we can say that *the increase in income is to the increase of number of sales as 1000:4* giving us a unit rate of \$250 in compensation per unit of cars sold. This is just the coefficient of  $N$  in the equation  $C = 1000 + 250N$ . To restate this: the number 250 expresses a relation between the variables  $N$  and  $C$ , even though the variables are not proportional. It is the *change in  $C$*  that is proportional to the *change in  $N$*  at the ratio 250:1.

Let's look at a few more examples to emphasize this point and to see how students should be able to extend this idea.

#### EXAMPLE 8.

At the statewide championship game, each player on each team receives five complimentary tickets, and can buy additional tickets at \$20 each. Carlos wants 8 tickets and Louis wants 16 tickets. How much does each pay for the full set of tickets?

**SOLUTION.** One might say that, since Louis is getting twice as many tickets, he has to pay twice as much. But that would be a mistake, the cost is not proportional to the number of tickets, but cost *is* proportional to the number of tickets *in excess of 5*. In this situation, they each get 5 complimentary tickets, so Carlos pays for 3 tickets and Louis pays for 11 tickets. At \$20 apiece, Carlos pays \$60 and Louis pays \$220.

By applying this thinking to the general case, we can write down a formula for the cost  $C$  of  $N$  tickets for any player. If a player wants  $N$  tickets, he gets 5 free and pays \$20 each for the remaining tickets. There are  $N - 5$  remaining, so the cost is  $C = 20(N - 5)$  or  $C = 20N - 100$ . The form of these equations tell different things, both interesting. The first ( $C = 20(N - 5)$ ) tells us that the cost is proportional to the excess of tickets above the first 5. The second tells us that the cost is \$20 per ticket, less \$100 for the free 5 tickets. Note that these equations make sense only for  $N \geq 5$ ; players don't get refunded if they have less than 5 friends. In figure 2 we have graphed this relationship:

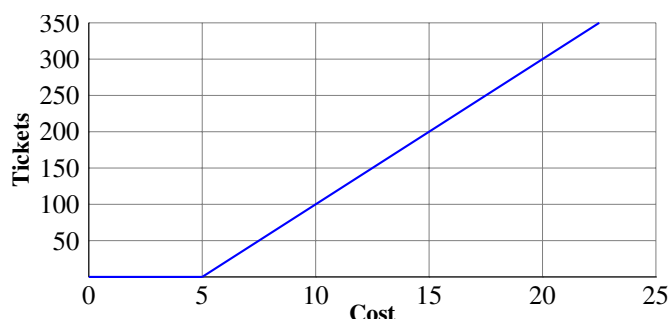


Figure 2

For any table where rate of change of one variable with respect to the other is **not** constant, the graph of these data will **not** be a straight line. Students will begin to explore these types of relationships more in Secondary 1. In 8th grade students simply need to recognize if a relationship is linear or not.

#### EXAMPLE 9.

I have a 120 gallon steel drum full of water to keep my garden thriving through a long dry spell. Each day I use four and a half gallons watering my plants. How much water do I have in the drum after 10 consecutive dry days? After  $d$  consecutive dry days? How long can I last without rain or refilling my drum?

**SOLUTION.** If I use 4.5 gallons of water each day, in 10 days, I use  $4.5(10) = 45$  gallons of water, so there are 75 gallons of water still in the drum. After  $d$  days there are still  $120 - 4.5d$  gallons in the drum. Using the symbol  $w$  to indicate the amount of water in the drum, this gives me the relation

$w = 120 - 4.5d$ . Figure 3 is the graph of that relation.

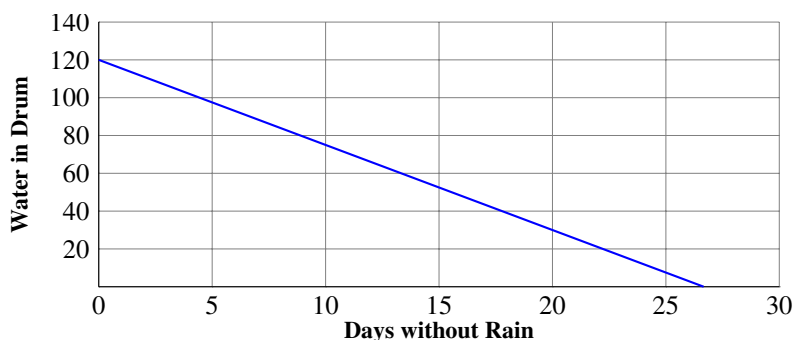


Figure 3

Notice that this time the line is pointing downward, that is because the amount of water in the drum decreases as the number of days increases. If we calculate  $(\text{change in } w)/(\text{change in } d)$  for any two points on the graph the result will be  $-4.5$ , indicating that each day we have 4.5 gallons less in the drum. It is important to note that language plays a role here: The word less accounts for the negative sign: it would be wrong to say that “each day we have  $-4.5$  gallons less in the drum.” What is correct is “the ratio of change in water to the change in day is  $-4.5$  gallons to 1 day.”

**EXAMPLE 10.**

Consider the image in Figure 4:

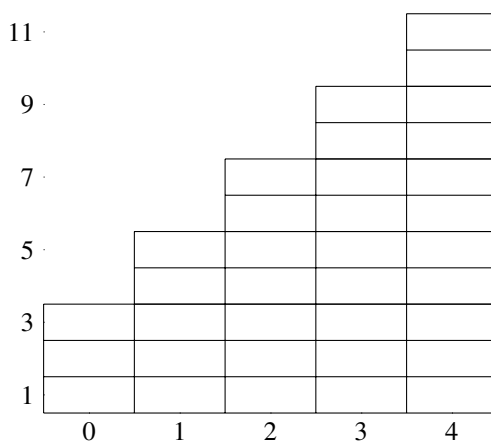


Figure 4

There is a pattern here: each time we move to the right by one unit, the height of the stack increases by 2. We have labeled the axes in figure 4 with  $x$  representing the number of moves to the right, and  $y$  the height of the stack. So, the first stack has the value 0, indicating that there are no moves to the right yet, and the last stack is 4 moves to the right. The height of the stack starts at 3, and with each move to the right, increases by 2. This tells us that the algebraic relationship is  $y = 3 + 2x$ .

**EXAMPLE 11.**

The Timpanooke trail (see the image) is a 7 mile trail from the foot of Mt. Timpanogos (at 7200 feet) to the peak (at 11900 feet). The trail has three different segments: the first is a three and a half mile horse trail with a steady altitude gain; the second is a two and a half mile traverse across a nearly level basin, and the last is a one mile steep climb to the peak. The accompanying table shows the altitudes at each of these transition points, and the time it takes an average hiker to cover each leg. Make two graphs; on both the horizontal axis is “miles” and on one, put “altitude” on the vertical axis, and on the other,

“hours.” Calculate, for each leg of the trek, the rate of change of altitude with respect to miles, and of hours with respect to miles. Compare and contrast the two graphical representations. Can you explain the similarities in the two graphs? This activity, of looking for similarities and differences among graphs will be studied in depth in Secondary 1.



[http://farm3.static.flickr.com/2659/3692964206\\_215e54c7d7.jpg](http://farm3.static.flickr.com/2659/3692964206_215e54c7d7.jpg)

Timpanooke Trail				
Altitude	7200	8700	10,700	11,900
Miles	0	3.5	6	7
Hours	0	2	3.5	5

In the above examples of linear relations, we have seen from the tables of values that the rate of change in  $y$  with respect to  $x$  is constant. That constant is positive if the graph points upwards as we move from left to right, and negative if the graph points downward. If the graph is horizontal, there is no change in  $y$ , so the rate of change is 0. It seems to always turn out that the graph of a linear relation is a straight line, but this is something we cannot yet explain. It is important to always keep in mind that the subject of mathematics - indeed every science - is not to just record observations, but to study them enough to be able to explain them. That is the only way that the working scientist can make progress, be it in mathematics, medicine or space travel. In classroom discussions, it is important, where relevant, to be precise about *observation* vs *understanding*. Here we're observing that the relationship is linear, we have not proven this fact.

So, what is the property of a line that guarantees that it will be described by a relation of the form  $y = mx + b$ ? What we know about a line is that it is determined by two points: place a straight edge against the two points and draw the line. A line is also determined by a point and a direction: lay the straightedge against the point, and set it in the intended direction and now draw the line. To relate these to a condition on linear relations, we need to find an algebraic way of expressing this geometric, constructive criterion. This is done with the concept of *slope* of a line and its relation to rate of change.

## Section 2.2. Slope of a Line

*Describe the effect of dilations ... on two dimensional figures using coordinates. 8.G.3: that the image of a line is a line parallel to it; that, under a dilation a line segment goes to a line segment whose length is the length of the original segment multiplied by the factor.*

*Use similar triangles to explain why the slope  $m$  is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation  $y = mx$  for a line through the origin and the equation  $y = mx + b$  for*

a line intercepting the vertical axis at  $b$ . 8.EE.6.

In order to respond to this last standard in the way it is stated, a chapter on transformational geometry up to similarity would have to precede this chapter. We felt that it is important in eighth grade to begin the year by completing the set of ideas around linearity, and that an initial chapter on geometry would be a diversion from this main point of 8th grade mathematics. Since all that is needed to understand the main fact about slope are the two properties of dilations cited above, we decided to minimize the geometry to these facts, and then return to the relation of the rate of change of a linear function and the slope of the graph of that function: they are the same. Dilations are connected to scaling, so it could be useful to recall at this time that discussion in 7th grade.

A *dilation* is given by a point  $C$ , the center of the dilation, and a positive number  $r$ , the factor of the dilation. The dilation with center  $C$  and factor  $r$  moves each point  $P$  to a point  $P'$  on the ray  $CP$  so that the ratio of the length of image to the length of original is  $r$ :  $|CP'|/|CP| = r$ .

Figure 5 illustrates a dilation. In the figure, the center of the dilation is  $C$ , and its factor is  $r$ . We have exhibited 3 original points,  $P, Q, R$  and their images under the dilation  $P', Q', R'$ . The letters  $a, b, c$  are the distances of  $P, Q, R$  from  $C$ .

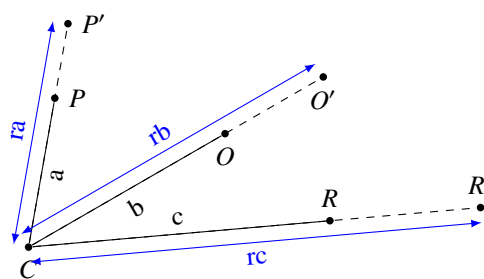


Figure 5

#### EXAMPLE 12.

In figure 6 we illustrate the effect of a dilation with center  $C = (0, 0)$ , and factor  $r = 2.5$  on a triangle in the first quadrant of a coordinate plane.

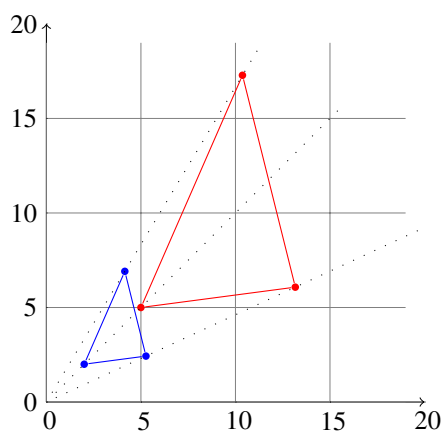


Figure 6

Observe the connection of this image with those in the 7th grade discussion of scale drawings. Note also that a point  $(x, 0)$  is moved to the point  $(2.5x, 0)$ , and a point  $(0, y)$  is moved to  $(0, 2.5y)$ . In fact, any point  $(x, y)$  is moved out to the point on the lines through the origin and that point whose distance from the origin is 2.5 times that of  $(x, y)$ . That the coordinates of this point is  $(2.5x, 2.5y)$  is easily observed, and gives the coordinate description of a dilation with center the origin. However, the understanding of why this is true needs the Pythagorean theorem, to which we will return in chapter 10. For now, students should work many examples of this type to conclude that

- In a coordinate plane, the dilation with center the origin and factor  $r$  is given by the coordinate rule  $(x, y) \rightarrow (rx, ry)$ .

Being able to express a transformation of the plane in terms of coordinates provides an algebraic tool to help work with conceptual understanding of dilations, but it is not as important at this stage as being able to understand the properties of dilations. Students should play with the concept sufficiently to accept these properties as intellectually plausible:

**Properties of the dilation with center  $C$  and factor  $r$ :**

- a. If  $P$  is moved to  $P'$ , then  $|CP'|/|CP| = r$ . That is, the distance of  $P'$  from  $C$  is  $r$  times the distance of  $P$  from  $C$
- b. If  $P$  is moved to  $P'$  and  $Q$  is moved to  $Q'$ , then  $|Q'P'|/|QP| = r$ . That is, under a dilation, the length of any line segment is multiplied by the factor of the dilation.
- c. The dilation takes parallel lines to parallel lines.
- d. A line and its image are parallel.

The first is part of the definition of a dilation. The second, that *every* length, not just those on lines through the center, is multiplied by the factor of the dilation, can be confirmed in examples - enough so that students accept this conclusion. The last two about parallelism are central properties of dilations. They are easy to observe through examples, and they are intuitively plausible. They follow from the fact that parallel lines do not intersect. As for c: if two lines do not intersect before the dilation, their images cannot intersect; this would imply that the dilation takes two different points to the same point; this is not possible. And for d, if a line and its image intersect, that point of intersection was not moved by the dilation. Unless  $r = 1$ , the only point not moved by the dilation is the center. In chapter 9 we will return to this subject and see that property b is a consequence of the other three properties in this statement.

In the preceding section we observed that the graph of a proportional relation is a straight line through the origin, we now turn to understanding why this statement and its converse is true. The key here is the above set of properties of dilations. Let's start with the statements that we want to understand:

- A non-vertical straight line through the origin is the graph of a proportional relation  $y = mx$ .
- The graph of the proportional relation  $y = mx$  is a non-vertical straight line through the origin.

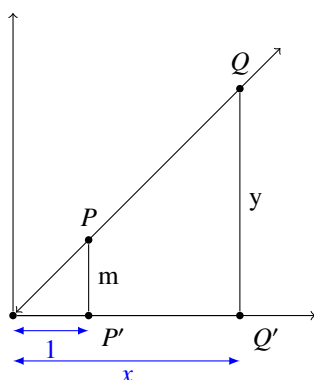


figure 7

We start with the first statement, and then show that it implies the second. In figure 7 we have drawn a typical line  $L$  through the origin.  $P$  is the point whose first coordinate is 1 and whose second coordinate is  $m$ .  $Q$  is any other point on the line with coordinates  $(x, y)$ . We introduce the dilation with center the origin that takes the point  $P'$  to  $Q'$ . Since the length 1 goes to the length  $x$ , the factor of the dilation is  $x$ . Now, the dilation takes the vertical line  $PP'$  to a parallel, and therefore also vertical line through  $Q'$ . That line has to intersect  $L$  at  $Q$ , since the line  $L$  is not changed in the dilation. Now, the dilation multiplies the length of  $PP'$  by  $x$ , so the length of  $QQ'$  is  $mx$ . But that is  $y$ , so we can conclude that  $y = mx$ . Since  $Q$  was any other point on  $L$ , we have shown that  $L$  is the graph of the proportional relationship  $y = mx$ .

As for the converse, we start with a proportional relationship  $y = mx$ . Draw the line through the origin and the point  $(1, m)$ . By the argument above, this line is the graph of the proportional relationship  $y = mx$ .

Now, we shall mimic this construction for the general non-vertical straight line. This less to we have similar



statements, with “proportional” replaced by “linear,” and “unit rate” replaced by “rate of change.” The statements we want to discuss are:

- A non-vertical straight line is the graph of a linear relation  $y = mx + b$ .
- The graph of the linear relation  $y = mx + b$  is a non-vertical straight line through the point  $(0, b)$  (called the  $y$ -intercept).

To see why these are true, start with a linear relation  $y = mx + b$ . Of course, when  $b = 0$ , the graph goes through the origin and is a proportional relationship, so, by the above argument the graph is a straight line. Now we could argue as follows: Given the equation  $y = mx + b$ , first look at the graph of the proportional relation  $y = mx$ . We now know that that is a straight line  $L$ . If we shift that graph in the vertical direction a distance of  $b$  units, we still have a straight line  $L'$ . We also know that if  $(x, y)$  are the coordinates of a point on  $L'$ , then  $(x, y - b)$  are the coordinates of a point on  $L$ . So we must have  $y - b = mx$ ; that is: this equation is satisfied by the coordinates of any point on  $L'$ . But this is the same as  $y = mx + b$ , so  $L'$  has to be the graph of the linear relation  $y = mx + b$ .

For the converse, start with a non-vertical line  $L$ . Since it is non-vertical it intersects the  $y$ -axis in a point  $(0, b)$ . If we shift this point to the origin, we get a new line  $L'$  through the origin which is, therefore, the graph of a proportional relationship  $y = mx$ . But if  $(x, y)$  is on  $L$ ,  $(x, y - b)$  is on  $L'$  and so we again have  $y - b = mx$  as a relation defining the line  $L$ , or what is the same  $y = mx + b$ .

This argument gives a geometric meaning to the number  $b$ : it is the  $y$ -coordinate of the point of intersection of the line with the  $y$ -axis (the  $y$ -intercept). But what is the geometric meaning of the number  $m$ ?

Let us start again with a non-vertical straight line in the coordinate plane, that doesn't go through the origin, but through some point  $(0, b)$  on the  $y$ -axis. We know that two points on a line determine the line: just put a straight edge against both points, and draw the pencil along the straightedge. We now want to see how to describe this in terms of coordinates: how do the coordinates of two points on a line determine the relation between the coordinates of any point on the line? This is where slope comes in.

Given two points in the coordinate plane,  $P$  and  $Q$ , we define the rise to be the difference of the  $y$  values from  $P$  to  $Q$ , and the run to be the difference in the  $x$  values from  $P$  to  $Q$ . The slope of the line segment is the quotient of these two differences:

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

If  $P$  has the coordinates  $(x_0, y_0)$  and  $Q$  has the coordinates  $(x_1, y_1)$  this is

$$\text{slope} = \frac{y_1 - y_0}{x_1 - x_0}$$

Geometrically, if we draw the triangle with hypotenuse the line segment from  $P$  to  $Q$  and legs horizontal and vertical - this is the *slope triangle* - the slope is the signed quotient of the length of the vertical leg by the length of the horizontal leg. By *signed*, we mean that the slope is positive if the line points upward as we go to the right, and negative if the line points downward. (see figures 8 and 9).

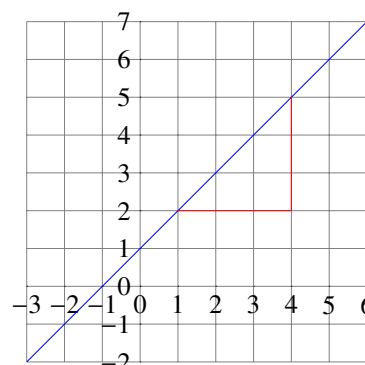


Figure 8

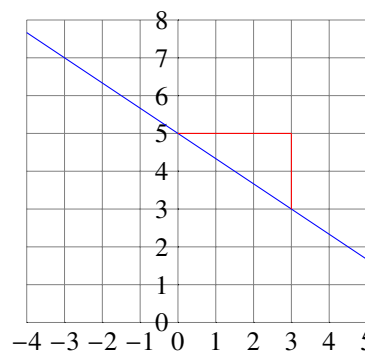


Figure 9

Note that in the slope computation the differences have to be taken in the same order: if we subtract the  $y$  value of  $P$  from that of  $Q$ , we must subtract the  $x$  value of  $P$  from that of  $Q$ . However, if we interchange the points  $P$  and  $Q$ , we get the same number. For a vertical line, the denominator in the quotient is zero, so the slope is not defined.

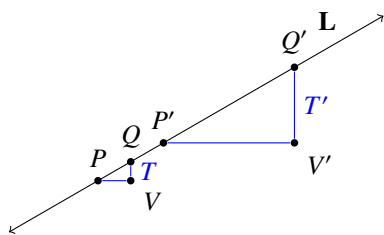


Figure 10

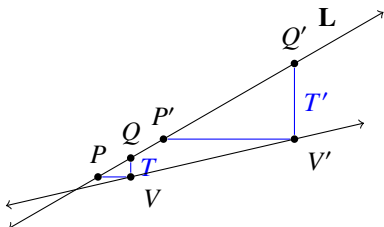


Figure 11

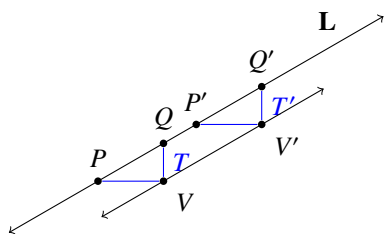


Figure 12

For a horizontal line, the numerator is zero, so, the slope is zero. Since the equation of a horizontal line is of the form  $y = b$ , this corresponds to the fact that  $y$  does not change as we move along the line. What we want to show is this: for a line  $L$ , this slope calculation is the same for any two points  $P$  and  $Q$  on  $L$  and is called the *slope of the line*.

Let  $L$  be a non-vertical line,  $P, Q$  and  $P', Q'$  two different pairs of points on the line, and  $T$  and  $T'$  the right triangles whose hypotenuses are the given line segments, and whose legs are horizontal and vertical. Label the vertices at the right angles as  $V$  and  $V'$  (see figure 10). These two triangles appear to be related by a dilation; we want to show that they are. First, if there is a dilation that takes  $T$  to  $T'$ , it must be the case that  $P$  goes to  $P'$ , so the line  $L$  is a line through the center of the dilation. Also,  $V$  goes to  $V'$ , so, by the same reasoning  $V$  and  $V'$  also lie on a line through the center of the dilation. Let  $L'$  be the line through  $V$  and  $V'$ . The point of intersection  $C$  of  $L$  and  $L'$  has to be the center of the dilation (figure 11), and its factor  $r$  has to be the ratio of the length of  $CP'$  to that of  $CP$ . Let's verify that this dilation does take  $T$  to  $T'$ . First of all, it takes  $P$  to  $P'$ , since that is how  $r$  was chosen. Since the dilation preserves "horizontal," and preserves the line  $L'$ , it takes the segment  $PV$  to  $P'V'$ , and so the ratio of those lengths is also  $r$ . Since the dilation preserves "vertical," and preserves the line  $L$ , it takes the segment  $QV$  to  $Q'V'$ , so the ratio of those lengths is also  $r$ . Thus, in moving from  $T$  to  $T'$ , the length of every side is multiplied by the same factor  $r$ , so when we calculate the *rise/run*, the  $r$ 's cancel, and the quotient is the same for both triangles.

There is one case not covered: the lines  $L$  and  $L'$  may not intersect; that is, they are parallel. In this case, (see Figure 12) under the shift of  $P$  to  $P'$ , the triangle  $T$  slides along these tracks to  $T'$  without changing the lengths of the sides.

### Section 2.3. The Equation $y = mx + b$ .

To wrap up this chapter, we bring together all the preceding material, not simply to summarize it, but also to lead in to the next chapter, a study of linear functions and lines, the purpose of which is to develop flexibility in moving among the representations of linear relations.

- For a line  $L$ , for any two points  $P, Q$  on the line, the quotient

$$\frac{\text{rise}}{\text{run}} = \frac{\text{change in } y \text{ from } P \text{ to } Q}{\text{change in } x \text{ from } P \text{ to } Q}$$

is constant, and that constant is the *slope of the line*.

#### EXAMPLE 13.

$(0, 5), (2, 9), (-1, 3)$  are three points on a line. Calculate rise/run for each pair of points.

**SOLUTION.** First we should verify that indeed the three points lie on a line, using the slope calculation. In each case that calculation produces 2 as the slope of the line, for example, taking the third and first points, we have:

$$\frac{3 - 5}{-1 - 0} = \frac{-2}{-1} = 2$$

Let us see what this example tells us. Start with two pairs of points; suppose that we find that the slope calculation produces the same number. This does not mean that the two pairs of points are on the same line (what does it mean?). However if there is a point in common to the two pairs, then all three do lie on the same line. This tells us something important. Given a line, pick two points  $P$ ,  $Q$  on the line. Calculate the slope  $m$  using these two points. Now take any point  $X$  on the plane, and calculate the slope of the slope triangle using the points  $X$  and  $P$  (or  $Q$ ). If the result is  $m$ , then, by this observation,  $X$  is on the line; if it is not  $m$ , then  $X$  is not on the line. We have generated a protocol for deciding whether or not a point  $X$  is on the line through  $P$  and  $Q$ .

Going back to example 12, the line through any pair of these points has slope 2. So, for any point  $(x, y)$ , if any of the calculations

$$\frac{y-5}{x-0}, \quad \frac{y-9}{x-3}, \quad \frac{y-3}{x-(-1)}$$

gives the value 2, then they all do, and  $(x, y)$  is a point on the line. If any of these computations do not give 2, then none do, and  $(x, y)$  is not on the line. So, we have this test for a point  $(x, y)$  to be on the line:

$$\frac{y-5}{x-0} = 2$$

By multiplying both sides by  $x$  we get  $-5 = 2x$ , or  $y = 2x + 5$ . This is called the equation of the line. Instead of choosing the first point, we could have chosen one of the other two getting the test:

$$\frac{y-9}{x-2} = 2, \quad \frac{y-3}{x-(-1)} = 2$$

No matter what point we choose for the test, after simplification we will always get the equation  $y = 2x + 5$ .

Chapter 3 starts with an examination of techniques to find the *equation of a line*, beginning with this example.