

Chapter 7

Rational and Irrational Numbers

In this chapter we first review the real line model for numbers, as discussed in Chapter 2 of seventh grade, by recalling how the integers and then the rational numbers are associated to points in the line. Having associated a point on the real line to every rational number, we ask the question, do all points correspond to a rational number? Recall that a point on the line is identified with the length of the line segment from the origin to that point (which is negative if the point is to the left of the origin). Through constructions (given by “tilted” squares), we make an observation first made by the Pythagorean society 2500 years ago that there are lengths (such as the diagonal of a square with side length 1) that do not correspond to a rational number. The construction produces numbers whose squares are integers; leading us to introduce the symbol \sqrt{A} to represent a number whose square is A . We also introduce the cube root $\sqrt[3]{V}$ to represent the side length of a cube whose volume is V . The technique of tilted squares provides an opportunity to observe the Pythagorean theorem: $a^2 + b^2 = c^2$, where a and b are the lengths of the legs of a right triangle, and c is the length of the hypotenuse.

In the next section we return to the construction of a square of area 2, and show that its side length ($\sqrt{2}$) cannot be equal to a fraction, so its length is not a rational number. We call such a number an *irrational number*. The same argument works for $\sqrt{5}$ and other lengths constructed by tilted squares. It is a fact that if N is a whole number, either it is a perfect square (the square of an integer), or \sqrt{N} is not a quotient of integers; that is \sqrt{N} is an *irrational number*.

In the next section we turn to the question: can we represent lengths that are not quotients of integers, somehow, by numbers? The ancient Greeks were not able to do this, due mostly to the lack of an appropriate system of expressing lengths by their numerical measure. For us today, this effective system is that of the decimal representation of numbers (reviewed in Chapter 1 of seventh grade).

We recall from grade 7 that a rational number is represented by a terminating decimal only if the denominator is a product of twos and fives. Thus many rational numbers (like $1/3, 1/7, 1/12, \dots$) are not represented by terminating decimals, but they are represented by repeating decimals, and similarly, repeating decimals represent rational numbers. We now view the decimal expansion of a number as providing an algorithm for getting as close as we please to its representing point on the line through repeated subdivisions by tenths. In fact, every decimal expansion represents a point on the line, and thus a number, and unless the decimal expansion is terminating or repeating, it is irrational.

The question now becomes: can we represent all lengths by decimal expansions? We start with square roots, and illustrate Newton’s method for approximating square roots: Start with some reasonable estimate, and follow with the recursion

$$a_{\text{new}} = \frac{1}{2} \left(a_{\text{old}} + \frac{N}{a_{\text{old}}} \right).$$

Through examples, we see that this method produces the decimal expansion of the square root of N to any required

degree of accuracy. Finally, we point out that to do arithmetic operations with irrational as well as rational numbers, we have to be careful: to get within a specified number of decimal points of accuracy we may need much better accuracy for the original numbers.

Section 7.1: Representing Numbers Geometrically

First, let us recall how to represent the rational number system by points on a line. With a straight edge, draw a horizontal line. Given any two points a and b on the line, we say that $a < b$ if a is to the left of b . The piece of the line between a and b is called the *interval between a and b* . It is important to notice that for two different points a and b we must have either $a < b$ or $b < a$. Also, recall that if $a < b$ we may also write this as $b > a$.

Pick a point on a horizontal line, mark it and call it the origin, denoted by 0. Now place a ruler with its left end at 0. Pick another point (this may be the 1 cm or 1 in point on the ruler) to the right of 0 and denote it as 1. We also say that the length of the interval between 0 and 1 is one (per *one unit*). Mark the same distance to the right of 1, and designate that endpoint as 2. Continuing on in this way we can associate to each positive integer a point on the line. Now mark off a succession of equally spaced points on the line that lie to the left of 0 and denote them consecutively as $-1, -2, -3, \dots$. In this way we can imagine all integers placed on the line.

We can associate a half integer to the midpoint of any interval; so that the midpoint of the interval between 3 and 4 is 3.5, and the midpoint of the interval between -7 and -6 is -6.5 . If we divide the unit interval into three equal parts, then the first part is a length corresponding to $1/3$, the first and second parts correspond to $2/3$, and indeed, for any integer p , by putting p copies end to end on the real line (on the right of the origin is $p > 0$, and on the left if $p < 0$), we get to the length representing $p/3$. We can replace 3 by any positive integer q , by constructing a length which is one q th of the unit interval. In this way we can identify every rational number p/q with a point on the horizontal line, to the left of the origin if p/q is negative, and to the right if positive.

The number line provides a concrete way to visualize the decimal expansion of a number. Given, say, a positive number a , there is an integer N such that $N \leq a < N + 1$. This N is called the *integral part* of a . If $N = a$, we are done. If not, divide the interval between N and $N + 1$ into ten equal parts, and let d_1 be the number of parts that fit in the interval between N and a . This d_1 is a *digit* (an integer between (and possibly one of) 0 and 9). This is the *tenths part* of a , and is written $N.d_1$. If $N.d_1 = a$, we are through. If not repeat the process: divide the interval between $N.d_1$ and $N.(d_1 + 1)$ into ten equal parts, and let d_2 be the number of these parts that fit between $N.d_1$ and a . This is the *hundredths part* of a , denoted $N.d_1d_2$. Continuing in this way, we discover an increasing sequence of numerical expressions of the form $N.d_1d_2\dots$ that get closer and closer to a , thus providing an effective procedure for approximating the number a . As we have seen in grade 7, this process may never (meaning “in a finite number of steps”) reach a , as is the case for $a = 1/3, 1/7$ and so on.

Now, instead of looking at this as a procedure to associate a decimal to a number, look at it as a procedure to associate a decimal to a length on the number line. Now let a be any point on the number line (say, a positive point). The same process associates a decimal expansion to a , meaning an effective way of approximating the length of the interval from 0 to a by a number of tenths of tenths of tenths (and so on) of the unit interval. We begin this discussion by examining lengths that can be geometrically constructed; leading us to an answer to the question: are there lengths that cannot be represented by a rational number? To do this, we need to move from the numerical representation of the line to the numerical representation of the plane.

Using the number line created above, draw a perpendicular (vertical) line through the origin and use the same procedure as above with the same unit interval. Now, to every pair of rational numbers (a, b) we can associate a point in the plane: Go along the horizontal (the x -axis) to the point a . Next, go a (directed) length b along the vertical line through a . This is the point (a, b) .

EXAMPLE 1.

In Figure 1 the unit lengths are half an inch each. To the nearest tenth (hundredth) of an inch, approximate the lengths AB , AC , BC .

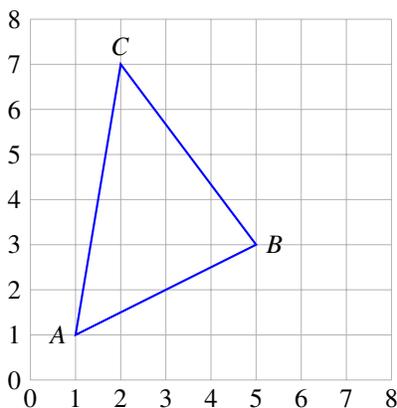


Figure 1

SOLUTION. We use a ruler to approximate the lengths. There will be a scale issue: the length of a side of a box may not measure half an inch on our ruler. After calculating the change of scale, one should find the length of AC to be about 3.05 in.

Now, it is important to know that, by using a ruler we can always estimate the length of a line segment by a fraction (a rational number), and the accuracy of the estimate depends upon the detail of our ruler. The question we now want to raise is this: can any length be described or named by a rational number?

The coordinate system on the plane provides us with the ability to assign lengths to line segments. Let us review this through a few examples.

EXAMPLE 2.

In Figure 2 we have drawn a tilted square (dashed sides) within a horizontal square. If each of the small squares bounded in a solid line is a unit square (the side length is one unit), then the area of the entire figure is $2 \times 2 = 4$ square units. The dashed, tilted square is composed of precisely half (in area) of each of the unit squares, since each of the triangles outside the tilted square corresponds to a triangle inside the tilted square. Thus the tilted square has area 2 square units. Since the area of a square is the square of the length of a side, the length of each dashed line is a number whose square is 2, denoted $\sqrt{2}$.

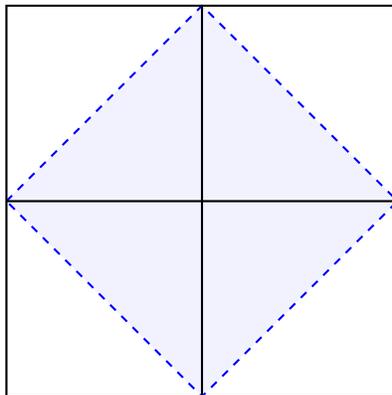


Figure 2

We will use this symbol \sqrt{A} (*square root*) to indicate a number a whose square is A : $a^2 = A$. Since the square of any nonzero number is positive (and $\sqrt{0} = 0$), \sqrt{A} makes sense only if A is not negative. Since $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, the integers 4, 9, 16, 25 have integers as square roots. A positive integer whose square root is a positive integer is called a *perfect square*. For other numbers (such as 2, 3, 5, 6, ...), we still have to find a way to calculate the square roots. This strategy, of tilted squares, gives a way of constructing lengths corresponding to square roots of all whole numbers, as we shall now illustrate.

EXAMPLE 3.

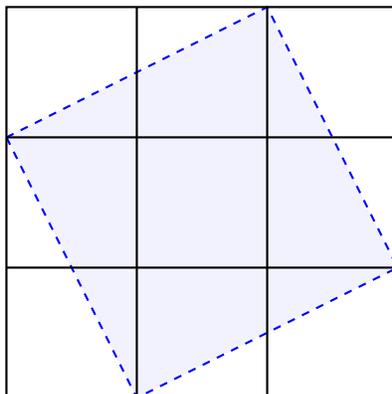


Figure 3

In Figure 3 the large square has side length 3 units, and thus area of 9 square units. Each of the triangles outside the tilted square is a 1×2 right triangle, so is of area 1. Thus the area of the tilted square is $9 - 4 = 5$, and the length of the sides of the tilted square is $\sqrt{5}$.

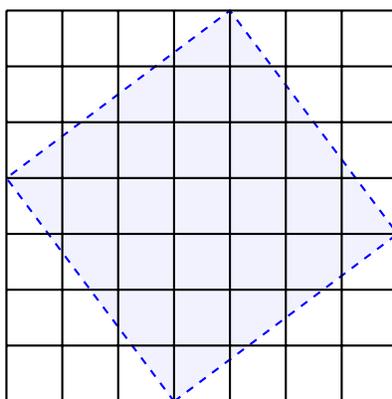


Figure 4

By the same reasoning: the large square in Figure 4 is 7×7 so has area 49 square units. Each triangle outside the tilted square is a right triangle of leg lengths 3 and 4, so has area 6 square units. Since there are four of these triangles, this accounts for 24 square units, and thus the area of the tilted triangle is $49 - 24 = 25$ square units. Since $25 = 5^2$, the side of the tilted square has length 5 units. That is, $\sqrt{25} = 5$.

Extension. Here we are using the “tilted square” construction in order to realize square roots of whole numbers as specific lengths, with the end goal of finding a way to represent those lengths numerically. (That is the long way of saying that eventually we will find a procedure to generate the decimal expansion of square roots). Some students may observe that the construction can be generalized to *any* initially given lengths a and b . They are right, and the generalization is the *Pythagorean Theorem* that relates the lengths of the sides of a right triangle. In Chapter 8, we will make this observation, and elaborate further on the Pythagorean theorem. The following example provides a preview of that discussion.

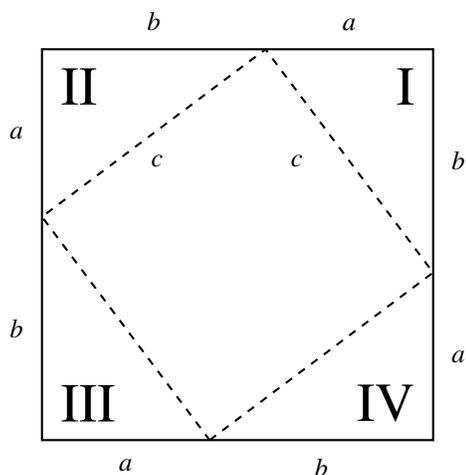


Figure 5

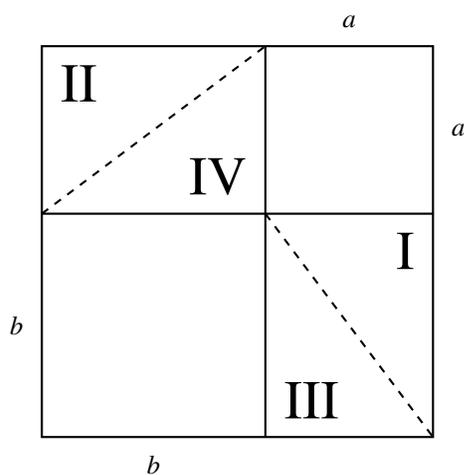


Figure 6

EXAMPLE 4.

For any two lengths a and b , draw a square of side length $a + b$ as shown in Figure 5. Now draw the dashed lines as shown in that figure.

The figure bounded by the dashed lines is a square; denote its side length by c . Then the area of the square bounded by the dashed lines is c^2 .

Each of the triangles in Figure 5 is a right triangle of leg lengths a and b and hypotenuse length c . Now, move to Figure 6 which subdivides the $a + b$ -sided square in a different configuration: the bottom left corner is filled with a square of side length b , and the upper right corner, by a square of side length a . The rest of the big square of Figure 6 is a pair of congruent rectangles. By drawing in the diagonals of those rectangles as shown, we see that this divides the two rectangles into four triangles, all congruent to the four triangles of Figure 5. Thus what lies outside these triangles in both figures has the same area. But what remains in Figure 5 is a square of area c^2 , and what remains in Figure 6 are the two squares of areas a^2 and b^2 . This result is:

The Pythagorean Theorem:

$$a^2 + b^2 = c^2$$

for a right triangle whose leg lengths are a and b and whose hypotenuse is of length c .

In this theorem we find the lengths of the sides of tilted squares algebraically. For example, the tilted square in Figure 1 has side length c where c is the length of the hypotenuse of a right triangle whose leg lengths are both 1:

$$c^2 = 1^2 + 1^2 = 2,$$

so $c = \sqrt{2}$. Similarly for Figure 2: $c^2 = 1^2 + 2^2 = 5$, so $c = \sqrt{5}$. For Figure 3, we calculate $c^2 = 3^2 + 4^2 = 9 + 16 = 25$, so $c = \sqrt{25} = 5$.

End Extension

Section 7.2: Solutions to Equations Using Square and Cube Roots

Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. 8.EE.2, first part.

In the preceding section we introduced the symbol \sqrt{A} to designate the length of a side of a square of area A . Similarly, the side length of a cube of volume V is $\sqrt[3]{V}$ (called the *cube root* of V). Another way of saying this is that \sqrt{A} is the solution of the equation $x^2 = A$ and $\sqrt[3]{V}$ is the solution of the equation $x^3 = V$. We also defined a perfect square as an integer whose square root is an integer. Similarly an integer whose cube root is an integer is a *perfect cube*. Here is a table of the first few perfect squares and cubes.

Number	1	2	3	4	5	6	7	8	9
Square	1	4	9	16	25	36	49	64	81
Cube	1	8	27	64	125	216	343	512	729

For numbers that are not perfect squares, we can sometimes use factorization to express the square root more simply.

EXAMPLE 5.

- Since 100 is the square of ten, we can write $\sqrt{100} = 10$. But we could also first factor 100 as the product of two perfect squares: $\sqrt{100} = \sqrt{4 \cdot 25} = \sqrt{4} \cdot \sqrt{25} = 2 \cdot 5 = 10$.
- Similarly, $\sqrt{729} = \sqrt{9 \cdot 81} = \sqrt{9} \cdot \sqrt{81} = 3 \cdot 9 = 27$. Also, $729 = 27^2$, and $3^3 = 27$, so $\sqrt[3]{729} = \sqrt[3]{27 \cdot 27} = \sqrt[3]{27} \cdot \sqrt[3]{27} = 3 \cdot 3 = 9$.
- Of course not every number is a perfect square, but we still may be able to simplify:
 $\sqrt{72} = \sqrt{36 \cdot 2} = \sqrt{36} \cdot \sqrt{2} = 6\sqrt{2}$.
- $\sqrt{32} = \sqrt{4 \cdot 4 \cdot 2} = 4\sqrt{2}$.

Similarly, we can try to simplify arithmetic operations with square roots:

- $\sqrt{6} \sqrt{12} = \sqrt{6} \sqrt{6 \cdot 2} = \sqrt{6} \sqrt{6} \sqrt{2} = 6\sqrt{2}$.
- $\sqrt{2} + \sqrt{8} = \sqrt{2} + \sqrt{4} \sqrt{2} = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$.

The following example goes beyond the eighth grade standards, but is included to emphasize to students that the rules of arithmetic extend to expressions with root symbols in them.

EXAMPLE 6.

- Solve: $3\sqrt{x} = 39$.

SOLUTION. Dividing both sides by 3, we have the equation $\sqrt{x} = 13$. Now squaring both sides, we get $x = 13^2 = 169$.

b. Solve: $5\sqrt{x} + \sqrt{x} = 24$.

SOLUTION. $5\sqrt{x} + \sqrt{x} = 6\sqrt{x}$, leading to the equation $6\sqrt{x} = 24$, simplifying to $\sqrt{x} = 4$. Squaring, we get the answer: $x = 4^2 = 16$.

c. Which is greater: 14 or $10\sqrt{2}$?

SOLUTION. We want to determine if the statement $14 < 10\sqrt{2}$ is true or false. We could approximate $\sqrt{2}$, but the truth of our answer depends upon the accuracy of our approximation. A better strategy is to square both sides, since we know that squaring two positive numbers preserves the relation between them. So our test becomes: is this true: $14^2 < 100(2)$. Since $14^2 = 196$, the answer is: yes, this is true.

d. Is there an integer between $4\sqrt{3}$ and $5\sqrt{2}$?

SOLUTION. To solve this, we need to square both sides and ask: is there a perfect square between these numbers. Now $(4\sqrt{3})^2 = 48$ and $(5\sqrt{2})^2 = 50$, and the only integer between these two is 49. Since $49 = 7^2$, the answer is “yes: 7”.

Section 7.3: Rational and Irrational Numbers

The Rational Number System

Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers, show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number. 8.NS.1.

Know that $\sqrt{2}$ is irrational. 8.EE.2, second part.

Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations. 8.NS.2

The relationship of numbers and lengths and their representation as decimals was a significant part of seventh grade mathematics. We now summarize that discussion.

The decimal representation of rational numbers is the natural extension of the base ten place-value representation of whole numbers. A decimal is a sequence of digits with a dot (called the *decimal point*) placed between a pair of consecutive digits. The sequence of the digits to the left of the dot is *the integral part* of the number and those to the right comprise the *fractional part*. Each place to the left of the dot represents a positive power of ten; each place to the right is a negative power of ten. The digit tells us how many of that power of ten are in the number. For example:

$$231.785 = 2(10^2) + 3(10) + 1 + 7(10^{-1}) + 8(10^{-2}) + 5(10^{-3}) = 2(100) + 3(10) + 1(1) + \frac{7}{10} + \frac{8}{100} + \frac{5}{1000}$$

If there is no fractional part, this may be indicated by placing a 0 after the decimal point (as in 24.0). If there is no whole number part in a given numeral, a 0 is usually placed before the decimal point (for example 0.75). There are only finitely many digits before the decimal point, but there may be infinitely many after the decimal point. In Grade 7 students discussed the decimal representation of a fraction; found by long division. Thus we recall, as examples:

$$\begin{aligned}\frac{3}{4} &= \frac{75}{100} = \frac{7}{10} + \frac{5}{100} = 0.75, \\ \frac{11}{4} &= 2\frac{3}{4} = 2 + \frac{75}{100} = 2 + \frac{7}{10} + \frac{5}{100} = 2.75, \\ \frac{2}{3} &= 0.66666666\cdots = 0.\overline{6}, \\ \frac{62}{11} &= 5.636363\cdots = 5.\overline{63}.\end{aligned}$$

In the two, the overbar represents the sequence of digits that are repeated over and over again, for as long as the student has the patience for long division.

A decimal is said to be *terminating* if it ends after finitely many steps in the long division, as in the case of the first two examples above. It is said to be eventually repeating if, after some number of places, there is a pattern of digits that repeats over and over again, as in the last two examples.

In Grade 7 students observed that the decimal expansion of a rational number always either terminates after a finite number of digits or eventually begins to repeat the same finite sequence of digits over and over. The *converse* is true: if a decimal terminates, or is eventually repeating, it describes a length corresponding to a rational number. We will now develop an understanding of why this is. First we start with terminating decimals.

EXAMPLE 7.

Convert 0.275 to a fraction.

SOLUTION.

$$0.275 = \frac{2}{10} + \frac{7}{100} + \frac{5}{1000} = \frac{2}{10} + \frac{7}{10^2} + \frac{5}{10^3}.$$

Now, if we put these terms over a common denominator, we get

$$\frac{2(10^2) + 7(10) + 5}{10^3} = \frac{275}{10^3} = \frac{275}{1000}.$$

In general, a terminating decimal is an integer plus a sum of fractions, all of whose denominators are powers of 10. By multiplying each term by 10/10 as many times as necessary, we can put all terms over the same denominator. We have shown that above and in the following. 0.67321 becomes

$$\frac{67321}{10^5} = \frac{67231}{100000},$$

and

$$0.0038 = \frac{3}{1000} + \frac{8}{10000} = \frac{38}{10000}.$$

We conclude that a terminating decimal leads to a fraction of the form $A/10^e$ where A is an integer and e is a positive integer. Notice that the expression $A/10^e$ is not necessary in lowest terms:

$$\frac{275}{10^3} = \frac{5 \cdot 5 \cdot 11}{2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5} = \frac{11}{2 \cdot 2 \cdot 2 \cdot 5} = \frac{11}{40}.$$

More generally, start with any decimal: represent it as a fraction with denominator a power of 10. As we reduce to lowest terms we can only cancel 2's and 5's in the denominator, leaving only factors of 2 and 5. **EXAMPLE 8.**

- a. $25 = 5^2$, so we should expect that $1/25$ can be represented by a terminating decimal. In fact, if we multiply by 1 in its disguise as $\frac{2 \times 2}{2 \times 2}$ we get:

$$\frac{1}{25} = \frac{4}{100} = 0.04 .$$

- b. Consider $1/200$. Since $200 = 2^3 \cdot 5^2$, we lack a factor of 5 in the denominator to have a power of 10. We fix this by multiplying the fraction by 1 in the form $\frac{5}{5}$ to get $5/1000 = .005$.

We conclude that a terminating decimal represents a fraction whose denominator is a product of 2's and 5's. This is the converse of what we learned in Grade 7. Thus

A terminating decimal leads to a fraction whose denominator is a product of 2's and/or 5's, and conversely, any such fraction is represented by a terminating decimal.

What about a fraction of the form p/q , where q is not a product of 2's and/or 5's? In Grade 6 students learn that by long division (of p by q) we can create a decimal expansion for p/q to as many places as we please. In Grade 7 this is taken a bit further. Since each step in the long division produces a whole number remainder that is less than q , after at most q steps we must repeat a remainder already seen. From that point on each digit of the long division repeats, and the process continues indefinitely in this way. For example, $1/3 = 0.3333\dots$ for as long as we care to continue the division.. For $10 \div 3$ produces a quotient of 3 with a remainder of 1, leading to a cyclical repeat of $10 \div 3$. In the way of review, let's look at a more complicated example.

EXAMPLE 9.

Find the decimal expansion of $157/660$.

SOLUTION. Dividing 157 by 660 gives a quotient of 0.2 and a remainder of 25. Divide 25 by 660 to get a quotient of 0.03 and a remainder of 52. So far we have

$$\frac{157}{660} = 0.23 + \frac{5.2}{660} .$$

$5.2 \div 660$ produces a quotient of 0.007 and a remainder whose numerator is 58. Now division by 660 gives a quotient of .0008 and a remainder whose numerator is 52. Since we've already seen that remainder, the long division will repeat itself and we'll again get a quotient of 8 and a remainder of 58. As this alternation of remainders of 52 and 58 continues forever, we can conclude that

$$\frac{157}{660} = 0.23787878\dots ,$$

with the sequence 78 repeating itself as often as we need. This is written as $157/660 = 0.23\overline{78}$, where the over line indicates continued repetition.

EXAMPLE 10.

Find the decimal expansion of $3/11$.

SOLUTION. Ignoring decimal points, the first division in the long division is $30 \div 11$, giving 2 with a remainder of 8. So, the second division is $80 \div 11$, giving 7 with a remainder of 3. Then the third division $30 \div 11$ gives a remainder of 8, which we've seen before. We have entered the repeating cycle with 27 as the repeating number. *Now* take care of the decimal point: $3/11$ is less than 1 and bigger than 0.1, so the answer is $3/11 = 0.\overline{27}$

These examples give ample evidence that the long division procedure will eventually go into a cycle. To summarize

The decimal expansion of a fraction is terminating or eventually repeating - that is, after some initial sequence of digits, there a following set of digits (which may consist of zeroes) that repeats over and over.

Express Decimals as Fractions

To complete this set of ideas we now demonstrate the converse of the boxed expression above: a terminating or eventually repeating decimal represents a fraction. We've already observed that a terminating decimal is a fraction whose denominator is a power of 10, so let's concentrate on repeating decimals.

EXAMPLE 11.

Start with: $0.33333\dots$, or in short notation $0.\overline{3}$. Let a represent this number. The effect of multiplying by 10 is to move the decimal to the right one digit. That is: $10a = 3.\overline{3}$ We now have the two equations:

$$a = 0.\overline{3}.$$

$$10a = 3.\overline{3}.$$

But recall the *meaning* of $3.\overline{3}$: it stands for $3 + 0.\overline{3}$ which is $3 + a$. So the second equation is $10a = 3 + a$. Solve for a to get $a = 3/9 = 1/3$.

Here's a more complicated example:

EXAMPLE 12.

Convert $0.\overline{234}$ to a decimal.

SOLUTION. Set $a = 0.\overline{234}$. Now multiply by 1000, to get these two equations:

$$a = 0.\overline{234}.$$

$$1000a = 234.\overline{234}.$$

The second is the same as: $1000a = 234 + \overline{.234}$; and becomes (after substitution) $1000a = 234 + a$, from which we conclude: $999a = 234$, so $a = 234/999$, which, in lowest terms, is $26/111$. Alternatively, we could subtract the first equation from the third, getting to $999a = 234$, and ultimately the same answer.

EXAMPLE 13.

Let's move on to decimals that are *eventually* repeating, such as $0.26\overline{54}$. First we take care of the repeating part: let $b = 0.\overline{54}$, and follow the method of the preceding examples to get the equation $100b = 54 + b$, so $b = 54/99 = 6/11$ in lowest terms. Now:

$$0.26\overline{54} = 0.26 + 0.00\overline{54} = \frac{26}{100} + \frac{1}{100}(0.\overline{54}) = \frac{26}{100} + \frac{1}{100}\left(\frac{6}{11}\right) = \frac{1}{100}\left(26 + \frac{6}{11}\right) = \frac{1}{100}\left(\frac{292}{11}\right) = \frac{73}{275}.$$

For a final example, let's convert $0.2\overline{3}$. We have:

$$0.2\overline{3} = \frac{2}{10} + \frac{1}{10}\left(\frac{1}{3}\right) = \frac{7}{30}.$$

Extension. Expand the Number System

In the preceding, for any point on the line, we have discussed how to get a sequence of terminating decimals that provide better and better estimates (in fact, each is within one-tenth of the given point than its predecessor). If we think of a point on the line as blur on the line (as it is in reality), then we can repeat this process until our decimal expansion is inside the blur, and then indistinguishable from the point of interest. Better optical devices will shrink the blur, and so we'll have to apply our process a few more times. Now, in the idealization of the real world that is the realm of mathematics, this process may never end; and so we envision infinite decimal expansions that can be terminated at any time to give us the estimate that is as close as we can perceive with our most advanced optics. Thus the repeating decimals make sense: if we want a decimal approximation of $1/3$ that is correct within one-thousandth, we take 0.3333 . If we want a decimal approximation correct within 500 decimal points, we take $0.$ followed by 500 3's.

Let us review the operational description of the decimal expansion through measurement: Let a be a point on the number line. Let N be the largest integer less than or equal to a ; that is $N \leq a < N + 1$. Now divide the interval between N and $N + 1$ into tenths, and let d_1 be the number of tenths between N and a . If this lands us right on a , then $a = N + d_1/10$. If not, divide the interval between $N + d_1/10$ and $N + (d_1 + 1)/10$ into hundredths and let d_2 be the number of hundredths below a , so that

$$N + \frac{d_1}{10} + \frac{d_2}{100} \leq a < N + \frac{d_1}{10} + \frac{d_2 + 1}{100}.$$

Do the same thing with thousandths and continue indefinitely - or at least as far as your measuring device can take you.

Some decimal expansions are neither terminating, nor ultimately repeating, for example

$$0.101001000100001000001 \dots,$$

where the number of 0's between 1's continues to increase by one each time, indefinitely. Another is obtained by writing down the sequence of positive integers right after each other:

$$0.123456789101112131415 \dots$$

Do such expressions really define (ideal) points on the line? This question, in some form or another, has been around as long as numbers have been used to measure lengths. Today, we know that it is not a question that can

be answered through common sense or logical deduction but, as an assertion that is fundamental to contemporary mathematics, must be accepted as a given (or, in some interpretations, as part of the definition of the line).

Such decimal expansions that are neither terminating or repeating cannot represent a rational number, so are said to be *irrational*. We can continue to make up decimal expansions that are neither terminating nor repeating, and in that way illustrate more irrational numbers. But what is important to understand is that there are constructible lengths (like the sides of some tilted squares) that are irrational.

End Extension

The fact that we are heading for is this: for any whole number N , if N is not a perfect square, then \sqrt{N} is not expressible as a fraction; that is, \sqrt{N} is an irrational number. The existence of irrational numbers was discovered by the ancient Greeks (about 5th century BCE), and they were terribly upset by the discovery, since it was a basic tenet of theirs that numbers and length measures of line segments were different representations of the same idea. What happened is that a member of the Pythagorean society showed that it is impossible to express the length of the side of the dashed square in Figure 2 ($\sqrt{2}$) by a fraction. Here we'll try to describe the modern argument that actually proves more:

If N is a whole number then either it is a perfect square or \sqrt{N} is not a fraction; that is, \sqrt{N} is *irrational*. Another way to say this is: For N a whole number, if \sqrt{N} is a fraction, then it is a whole number.

Another way of making this statement is this: If N is a whole number, and \sqrt{N} is rational, then \sqrt{N} is also a whole number.

Extension. The Greeks knew how to show this for $N = 2$. Euclid gave a geometric proof (based on the concept of *commensurable lengths*, and Aristotle gave an algebraic proof, as follows:

Suppose that $\sqrt{2}$ is a rational number. Express it as $\sqrt{2} = \frac{p}{q}$ in lowest terms (meaning that p and q have no common integer factor). In particular, it cannot be the case that p and q are both even. Now square both sides of the equation to get

$$2 = \frac{p^2}{q^2} \quad \text{so that} \quad 2q^2 = p^2 .$$

This last equation tells us that p^2 is even (it has 2 as a factor). But, since the square of an odd number is odd, that tells us that p must be even: $p = 2r$ for some integer r . Then $p^2 = 4r^2$, and putting that in the last equation above we get

$$2q^2 = 4r^2 \quad \text{so that} \quad q^2 = 2r^2 ,$$

and q is thus also even. This contradicts the statement that $2 = p/q$ in lowest terms, so we conclude that our original assumption that $\sqrt{2}$ is equal to a ratio of integers was false. Therefore, since $\sqrt{2}$ can not be represented as a ratio of integers, $\sqrt{2}$ is irrational.

The argument for any whole number N (that it is either a perfect square or irrational) is similar, but uses more about the structure of whole numbers (in terms of primes), It might be interesting to go into this, but it is beyond the 8th grade core.

EXAMPLE 14.

Show that $\sqrt{50}$ is irrational.

SOLUTION. $50 > 7^2$ and $50 < 8^2$, so $\sqrt{50}$ lies between 7 and 8, and thus cannot be an integer. Another way to see this is to factor 50 as $25 \cdot 2$, from which we conclude that $\sqrt{50} = 5\sqrt{2}$, and $\sqrt{2}$ is irrational.

EXAMPLE 15.

π is another irrational number that is defined geometrically. How do we give a numerical value to this number? The ancient Greek mathematician, Archimedes, was able to provide a geometric way of approximating the value of π . Here we describe it briefly, using its definition as the quotient of the circumference of a circle by its diameter. In Figure 7, consider the triangles drawn to be continued around the circle 8 times.

Now measure the lengths of the diameter (5.3 cm) and the lengths of the outer edge of the triangles shown triangles. The length of the inner one is 2 cm, and of the outer one is 2.3 cm. Now, reproducing the triangles in each half quadrant, we obtain two octagons, one with perimeter 8×2 , which is less than the circumference of the circle and the other with perimeter 8×2.3 which is greater than the circumference of the circle. Since we know the formula $C = \pi D$, where C is the circumference of a circle, and D its diameter, we obtain

$$8 \times 2 < C = \pi D = 5.3\pi \quad \text{and} \quad 8 \times 2.3 > C = \pi D = 5.3\pi .$$

After division by 5.3, these two inequalities give the estimate: $3.02 < \pi < 3.47$. To increase the accuracy, we increase the number of sides of the approximating polygon. Archimedes created an algorithm to calculate the polygon circumferences each time the number of sides is doubled. At the next step (using a polygon with 16 sides), the procedure produces the estimate $22/7$ for π , which has an error less than 0.002.

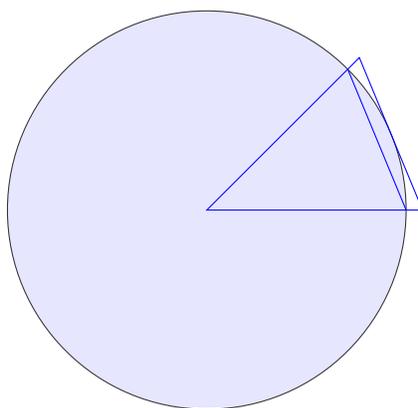


Figure 7

End Extension

Approximating the Value of Irrational Numbers

Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). 8.NS.2

Since irrational numbers are represented by decimals that are neither terminating nor repeating, we have to rely on the definition of the number to find fractions that are close to the irrational number - hopefully as close as we need the approximation to be. So, as we saw above, Archimedes used the definition of π to find a rational number ($22/7$) that is within $1/500$ of π . Even better, Archimedes described an algorithm to get estimates that are closer and closer - as far as we need them to be. But how about square roots?

We know that $\sqrt{2}$ can be represented as the diagonal of a right triangle with leg lengths equal to 1, so we can measure the length of that diagonal with a ruler. But the accuracy of that measure depends upon the detail in our ruler. We'd rather have an arithmetic way to find square roots - or, to be more accurate, to approximate them. For this, we return to the discussion of decimals in 7th grade. There, a method was described that found decimals that came as close as possible to representing a given length. However, it too, depended upon our capacity to measure. What we want is a method to approximate square roots, which, if repeated over and over again, gives us an estimate of the square root that comes as close to the exact length as we need it to be.

Before calculators were available, a method for estimation of square roots was that of trial and error. With calculators, it is not so tedious - let us describe it.

EXAMPLE 16.

Approximate $\sqrt{2}$ correct to three decimal places.

SOLUTION. Since $1^2 = 1$ and $2^2 = 4$, we know that $\sqrt{2}$ is between 1 and 2. Let's try 1.5. $1.5^2 = 2.25$, so 1.5 is too big. Now $1.4^2 = 1.96$ so 1.4 is too small, but not by much: 1.4 is correct to one decimal place. Let's try 1.41: $1.41^2 = 1.9881$ and $1.42^2 = 2.0164$. It looks like 1.42^2 is a little closer, so let's try 1.416: $1.416^2 = 2.005$; so we try $1.415^2 = 2.002$, still a little large, but quite close. To see that we are within three decimal places, we check $1.414^2 = 1.9994$. That is pretty close, and less than 2. To check that 1.414 is correct to three decimal places, we check half an additional point upwards: $1.4145^2 = 2.0008$, so the exact value of $\sqrt{2}$ is between 1.4140 and 1.4145, so 1.414 is correct to three decimal places.

EXAMPLE 17.

Gregory wants to put a fence around a square plot of land behind his house of 2000 sq.ft. How many linear feet of fence will he need?

SOLUTION. His daughter, a member of this class, tells him that the length of a side will be $\sqrt{2000}$, and since there are four sides, Gregory will need $4\sqrt{2000}$ linear feet of fence. He objects that he can't go to the lumber yard asking for $4\sqrt{2000}$ linear feet of fence; he needs a number. His daughter thinks, well that is a number, but understands that he needs a decimal approximation. She says: "First we try to get close: The square of 30 is 900; so we try 40: its square is 1600, and $50^2 = 2500$, so our answer is between 40 and 50. Let's try 45; since $45^2 = 2025$, we are close." Gregory says that for his purpose, this is close enough, but he is curious and asks his daughter if she can do better. "Sure," she says, and calculates $44^2 = 1936$, and informs her father that the answer is between 44 and 45 - and maybe closer to 45. So she calculates the squares of number between 45.5 and 50:

Number	Square of number
44.5	1980.25
44.6	1989.16
44.7	1998.09
44.8	2007.04

and concludes that each plot has to have side length between 44.7 feet and 44.8 feet, and much closer to 44.7. She suggests that he go with 44.72; that is he will need $4 \times 44.72 = 178.88$ linear feet of fence. Checking, she calculates that the square of 44.72 (accurate to two decimal points) is 1999.88.

EXAMPLE 18.

Find the square root of 187 accurate to two decimal places.

SOLUTION. Since $10^2 = 100$, we try $11^2 = 121$, $12^2 = 144$, $13^2 = 169$ and $14^2 = 196$. From this we conclude that $\sqrt{187}$ is between 13 and 14, and probably a little closer to 14. Next, try 13.7: $13.7^2 = 187.69$, a little too big. Now try 13.65: $13.65^2 = 187.005$; almost there! Now, to be sure we are correct to 2 decimal points, we calculate $13.64^2 = 186.05$. Since 13.65^2 is so much closer to 187, we conclude that 13.65 is the answer, accurate to two decimal places.

Trial and error seems to work fairly well, so long as we start with a good guess, and have a calculator at hand. But, in the field, the engineer will want a way to do this with a calculator or computer, where the computer does the guessing. Such a method is called “Newton’s method,” after Isaac Newton, one of the discoverers of the Calculus. Newton reasoned this way: suppose that we have an estimate a for the square root of N , That is: $a^2 \sim N$, where the symbol \sim means “is close to.” Dividing by a , we have $a \sim N/a$, so N/a is another estimate, about as good as a . He also noticed that a and N/a lie on opposite sides of \sqrt{N} : as follows: suppose $a < \sqrt{N}$, so that $a^2 < N$. Multiply both sides by N , giving $Na^2 < N^2$. Now divide both sides by a^2 to get:

$$N < \frac{N^2}{a^2} \quad \text{or} \quad \sqrt{N} < \frac{N}{a}.$$

Because of this, the average of a and N/a average should be an even better estimate . So he set

$$a' = \frac{1}{2} \left(a + \frac{N}{a} \right),$$

and then repeated the logic with a' , and once again with the new estimate, until the operation of taking a new average produced the same answer, up to the desired number of decimal places. The amazing fact is that Newton showed that this actually works and in a small number of steps, and can be applied to find solutions of a wide range of numeric problems. Let’s now illustrate inf the following example.

EXAMPLE 19.

Find the square root of two accurate up to four decimal places.

SOLUTION. We try 1 and 2, getting $1^2 = 1$ and $2^2 = 4$. Now, $2/1 = 2$, so according to Newton, we should now try the average: 1.5. Since $2/(1.5) = 4/3$, we next try the new average:

$$a' = \frac{1}{2} \left(1.5 + \frac{4}{3} \right) = \frac{17}{12} = 1.41\bar{6}.$$

This already is a pretty good estimate.

Extension. Now, let us bring in calculators or a computational program like Excel so that we can repeat this process until the new estimate is the same as the second. Excel calculations for $\sqrt{2}$ (using Newton’s method) are shown in the table of Figure 8. Our discussion brought us to the second line (1.416666667 is 17/6 correct to 9 decimal places). The table continues with Newton’s method, until the estimate (last column) stabilizes. Actually, it stabilized (to four decimal places) in the third line, but we have continued the calculation, to show that there is

no further change. In fact each step in Newton's method produces a more accurate estimate, so once we have no change (to the number of correct decimals we require), there is no need to go further.

First estimate: $a = 1$

Estimate	$2/a$	New Estimate (Average)
1	2	1.5
1.5	1.333333	1.41666667
1.41667	1.411765	1.414215686
1.41422	1.414211	1.414213562
1.41421	1.414214	1.414213562
1.41421	1.414214	1.414213562
1.41421	1.414214	1.414213562
1.41421	1.414214	1.414213562

Figure 8: Square Root of 2

EXAMPLE 20.

Use Newton's method to find the square root of 5.

SOLUTION. First of all "find the square root of five" is not very meaningful. Using the tilted square with side lengths 1 and 2, we found the square root of 5 as a length. So, perhaps here we mean, "find the numerical value of the square root of 5." But as we have already observed, $\sqrt{5}$ is not expressible as a fraction, so we can't expect to "find" its value precisely. What we can hope for is to find a decimal expansion that comes as close as we please to $\sqrt{5}$. So, let us make the question precise: find a decimal that is an estimate of $\sqrt{5}$ that is correct to 4 decimal places. Let's go through Newton's method.

First we see that 2 is a good approximation for $\sqrt{5}$ by an integer, since $2^2 = 4$ and $3^2 = 9$. Now $5/2 = 2.5$ is also a good guess, since $2.5^2 = 6.25$. In fact it is what Newton's method wants us to look at next. The rest of the computation was done by Excel and is shown in Figure 8a. There we see that by the third step in Newton's method, we have $\sqrt{5}$ correct to 6 decimal places.

First estimate: $a = 2$

Estimate	$5/a$	New Estimate (Average)
2	2.5	2.25
2.25	2.222222	2.236111111
2.23611	2.236025	2.236067978
2.23607	2.236068	2.236067977
2.23607	2.236068	2.236067977
2.23607	2.236068	2.236067977
2.23607	2.236068	2.236067977
2.23607	2.236068	2.236067977

Figure 9: Square Root of 5

Do we ever get to the place where the "last two approximations" agree perfectly - that is: when have we arrived at the numerical value of the desired point? Alas, for irrational numbers, the answer is "we never do, but we do get better and better." Unless N itself is a perfect square (the square of another integer), we will never arrive at a decimal expansion that is precisely \sqrt{N} . But - and this is all we really need - we can get as close as we want.

EXAMPLE 21.

Now use Newton's method to calculate $\sqrt{25}$ and $\sqrt{1000}$.

SOLUTION. The calculation is exhibited in Figure 10a and 10b respectively.

First estimate: $a = 1$

Estimate	$25/a$	New Estimate (Average)
1	25	13
13	1.923077	7.461538462
7.46154	3.350515	5.406026963
5.40603	4.624468	5.015247602
5.01525	4.984799	5.000023178
5.00002	4.999977	5
5	5	5

Figure 10a: Square Root of 25

First estimate: $a = 25$

Estimate	$1000/a$	New Estimate (Average)
25	40	32.5
32.5	30.76923	31.63461538
31.6346	31.61094	31.62277882
31.6228	31.62277	31.6227766
31.6228	31.62278	31.6227766
31.6228	31.62278	31.6227766
31.6228	31.62278	31.6227766
31.6228	31.62278	31.6227766
31.6228	31.62278	31.6227766

Figure 10b: Square Root of 1000

End Extension.

EXAMPLE 22.

Find a simpler algebraic expression for $\sqrt{1000}$.

SOLUTION. To be clear, we are asked to factor 1000 in such a way so as to pull out as large a perfect square as possible. Since $1000 = 100 \cdot 10$, and 100 is a perfect square, the answer is

$$\sqrt{1000} = 10\sqrt{10} = 10\sqrt{2}\sqrt{5}.$$

We could argue that, to approximate $\sqrt{1000}$, instead of applying a square root method to 1000, we could use the approximations of $\sqrt{2}$ and $\sqrt{5}$ that we already have and multiply. The approximations we already have (see Figures 8 and 9) are: $\sqrt{2} = 1.414213562$ and $\sqrt{5} = 2.236067977$. Multiplying, we find:

$$\sqrt{1000} = 10\sqrt{2}\sqrt{5} = (10)(1.414213562)(2.236067977) = 31.6227752.$$

Now, compare this with Figure 10b and note the loss of accuracy in the sixth and seventh decimal places. This is not an error in multiplication (which was done on a basic hand calculator), but a fact about multiplying decimal approximations: always expect a loss of accuracy in the last decimal places.

The closeness of approximations changes as we add or multiply them. So, if A' is a 2-decimal approximation to A and B' is the same for B , we cannot conclude that $A' + B'$ is a 2-decimal approximation to $A + B$, nor that $A'B'$ is a 2-decimal approximation to AB . Let us illustrate that.

EXAMPLE 23.

3.16 agrees with $\sqrt{10}$ to two decimal places. But $3.16 \times 3.16 = 9.9856$, which does not agree with 10 to two decimal places.

EXAMPLE 24.

Approximate $\pi + \sqrt{2}$ to three decimal places. Start with the approximations up to four decimal points: 3.1416 and 1.4142, and add: $3.1416 + 1.4142 = 4.5558$, from which we accept 4.556 as the three (not four) decimal approximation because of the rounding to the fourth place in the original approximations.

For products it is not so easy. Suppose that A' and B' are approximations to two particular numbers, which we can denote as A, B . Then we have $A = A' + e_1$, $B = B' + e'$ where e and e' are the errors in approximation. For the product we will have $AB = (A' + e)(B' + e') = A'B' + A'e' + B'e + ee'$, telling us that the magnitude of the error has been multiplied by the factors A and B , which may put us very far from the desired degree of accuracy.

EXAMPLE 25.

Suppose that A and B approximate the numbers 1 and 100 respectively within one decimal point. That means that

$$0.95 < A < 1.05 \quad \text{and} \quad 99.95 < B < 100.05 .$$

When we multiply, we find that AB could anywhere in the region

$$94.9525 < AB < 105.0525 ,$$

thus, as much as 5 units away from the accurate product 100.

To find approximate values for irrational numbers, we have to understand the definition of the number so we can use it for this purpose. For example, suppose we want to find a number whose cube is 35, correct up to two decimal places. Start with a good guess. Since $3^3 = 27$ and $4^3 = 64$, we take 3 as our first guess. Since 27 is much closer to 35 than 64, we now calculate the cubes of 3.1, 3.2, 3.3, ... until we find those closest to 35: $3.1^3 = 29.791$, $3.2^3 = 32.6768$, $3.3^3 = 35.937$. We can stop here, since the last calculated number is larger than 35. Now we go to the next decimal place, starting at 3.3, working down (since 35.937 is closer to 35 than 32.6768: $3.29^3 = 35.611$, $3.28^3 = 35.287$, $3.27^3 = 34.965$. Since the last number is as close as we can get to 35 with two place decimals, we conclude that 3.27 is correct to two decimal places.

EXAMPLE 26.

What is bigger π^2 or 10?

SOLUTION. In the days before computers, the high school physics teacher recommended to students that they could use 10 as an approximation to π^2 so as to simplify some calculations when necessary. Here we ask: which is bigger? Today we use $22/7$ as an approximation to π , so let's try $(22/7)^2$. That is equal to 9.877 which is sufficiently less than 10, so we can suspect that $\pi^2 < 10$. Now, recall from example 3 that $3.16^2 = 9.9856$. Since we know that $3.16 > \frac{22}{7}$, we have stronger evidence that $\pi^2 < 10$. In fact, a seven-decimal approximation to π is 3.141529, so we can conclude that $\pi < \frac{22}{7} < 3.16$ and thus $\pi^2 < 10$