Chapter 4
Analyze Proportional Relationships and Use Them to Solve Real-World Problems

Because of their prevalence and importance in science and every-day life, it is important that students understand ratios and proportional relationships so they become informed consumers and critical thinkers. This chapter focuses on that understanding starting with the concept of emphunit rate as introduced in sixth grade. Two related quantities A and B are said to be in the *ratio relationship* $a : b$ if the number of elements in A divided by the number of elements in B is a fraction equivalent to a/b. The quotient a/b is the *unit rate* of A with respect to B, meaning for every unit of B there are a/b units of A. For example, if I can walk 2 laps around the track every 10 minutes, the unit rate is 2/10 = 1/5 laps per minute.

In 7th grade we introduce the concept of a *proportional relationship* between two quantities, and relate it to the idea of ratio. In order to develop a broad understanding, we use a variety of approaches, including bar models, tables, graphs, and equations. Students will use these tools to determine when two quantities are proportional. Working with proportional relationships allows one to solve many real-life problems such as adjusting a recipe, quantifying chance (odds and probability), scaling a diagram (drafting and architecture), and finding percent increase or percent decrease (price markup, discount, and tips). The chapter begins with an anchor problem about lemonade recipes which raises important ideas of unit rates, ratio and proportion.

The study of ratios and proportional relationships extends students' work with multiplication, division and measurement from earlier grades and forms a strong foundation for further study in mathematics and science. Later in this course, students will use proportions to solve scaling problems, including making scale drawings. In 8th grade, proportions form the basis for understanding the concept of constant rate of change (slope) and students learn that proportional relationships are a subset of linear relationships, and thus are represented by linear functions. As students progress, they use ratios in algebra (functions), trigonometry (the basic trigonometric functions) and calculus (average and instantaneous rate of change of a function). An understanding of ratio is essential in the sciences to make sense of speed, acceleration, density, surface tension, electric or magnetic field strength, and strength of chemical solutions. Ratios also appear in descriptive statistics, including demographic, economic, medical, meteorological, and agricultural statistics (e.g. birth rate, per capita income, body mass index, rain fall, and crop yield). Ratios underlie a variety of measures, for example, in finance (exchange rate), medicine (dose for a given body weight), and technology (kilobits per second).

Section 4.1: Understand and Apply Unit Rates

Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. 7.RP.1

In 6th grade, students learned that a ratio is a comparison of the size of two quantities. An example is the relation between the number of boys and girls in the class. One possible instance would be a class of 28 students composed of 16 girls and 12 boys. The ratio could be looked at in several ways: *part-to-part* as in 16 girls : 12 boys or *part-to-whole* as in 16 girls : 28 students. Another class would be said to be *in the same ratio* as this class if the girl:boy ratio is the same. So a class with 12 girls and 9 boys is in the same ration as the given class, but another class with 15 girls and 12 boys is not. This section will briefly review concepts from 6th grade and extend them to proportions and unit rates with the goal of solving real-world problems. Students connect and build on this knowledge by first modeling situations with tape diagrams or bar models, illustrated in the first examples.

Example 1.

A chocolate chip cookie recipe calls for three cups of flour and two cups of sugar. Use a tape diagram (bar model) to represent this situation.

Solution.

1 cup flour	1 cup flour	1 cup flour	1 cup sugar	1 cup sugar

Example 2.

If Alex only has one cup of sugar, how much flour should he use in making the cookies?

Solution. Alex notices that he has half the sugar needed for the recipe, and therefore can only make half a recipe. He copies the above tape, but changes units to half what they were in example 1.

1/2 cup flour	1/2 cup flour	1/2 cup flour	1/2 cup sugar	1/2 cup sugar

Now Alex observes that this formulation shows the amount of sugar he has available, and so he will need 3 half cups of flour; that is 1.5 cups of flour.

Example 3.

Alex wants to make many cookies to bring to a party so he buys more sugar. If he uses six cups of sugar how much flour will he need?

Solution. Following the same logic, Alex can repeat the first tape above but now the units are "3 cups." Alternatively he can reason that he will now have 3 times the sugar needed for the first tape recipe, so all he has to do is repeat that tape 3 times.

1 cup flour	1 cup flour	1 cup flour	1 cup sugar	1 cup sugar
1 cup flour	1 cup flour	1 cup flour	1 cup sugar	1 cup sugar
1 cup flour	1 cup flour	1 cup flour	1 cup sugar	1 cup sugar

Example 4.

Riley has a job helping the neighbor with yardwork. One day Riley worked for two hours and earned \$11.00. How much does she earn each hour?

Solution. Unlike cups of flour and cups of sugar, dollars and hours are different units of measure. We create a comparison tape diagram by drawing two bars of equal length, one on top of the other, to represent the situation. We model each quantity in its own bar.

2 hours
\$11.00

Since we are looking for the amount Riley makes each hour, and there are two hours, divide each bar into two equal pieces. This gives 1 hour in each piece of the top bar and \$5.50 in each of the bottom bars. Therefore, Riley makes \$5.50 per hour helping the neighbor.

1 hour	1 hour
\$5.50	\$5.50

1 hour						1 hour				
\$ 1	\$1	\$1	\$ 1	\$1	\$1	\$ 1	\$1	\$1	\$1	\$1

Example 5.

Riley wants to make a table of how much money she'll earn by working 0, 1, 2, 3, 4, and 5 hours.
Solution. The tape diagram can again be iterated to model the situation.

1 hour	1 hour	1 hour	1 hour	1 hour
\$5.50	\$5.50	\$5.50	\$5.50	\$5.50

Hours	0	1	2	3	4	5
Dollars	0	5.50	11.00	16.50	22.00	27.50

Notice the patterns in the table from the previous example. Each time the number of hours increases by one, the number of dollars increases by 5.50. If you double the number of hours worked from two to four, the amount of money earned also doubles from \$11.00 to \$22.00. Also note that the number of dollars is equal to the number of hours multiplied by 5.50. Often it is the case that different instances of two quantities lead to **equivalent ratios**.

> Two ratios $a : b$ and $c : d$ are **equivalent** if there is a number p such that $c = ap$ and $d = bp$. If the ratios are considered in fractional form, the ratio $a : b$ is equivalent to the ratio $c : d$ if $\frac{c}{d} = \frac{ap}{bp}$ where denominators are nonzero.

Note that the notion of equivalence of ratios is the same as the equivalence of fractions, and the fraction is the unit rate: $a : b$ is equivalent as a ratio to $c : d$ precisely when the fractions a/b and c/d are equivalent.

In the cookie examples, the ratios of flour to sugar are expressed in the different scenarios as 3:2, (3/2):1, and 9:6. Since $3 = (3/2) \cdot 2$, $2 = 1 \cdot 2$, $9 = (3/2) \cdot 6$, and $6 = 1 \cdot 6$, the ratios are equivalent. Similarly, in Riley's example, the nonzero ratios of dollars to hours are all equivalent to 5.5/1. The ratios 1.5:1 and 5.5/1 are important because they tell us how many of the first quantity for one unit of the second quantity, the

unit rate. In our examples, the unit rate of cups of flour to cups of sugar is 1.5 and the unit rate of dollars to hours is 5.5. In general, if we know the amount of y corresponding to one unit of x, then we can compute the amount of y for any quantity of x. Therefore, if there are r units of y for every one unit of x, then for m units of x, y will be rm. This number r is the unit rate of y with respect to x.

Unit rates can be computed with either of the two quantities as the basic unit. Using the models in the examples above, we can compute the unit rate of sugar to flour is 2/3 : 1 and hours to dollar is 2/11 : 1. Note that in fractional form, the unit rate of flour to sugar is 3/2 and the unit rate of sugar to flour is 2/3. Similarly, the unit rate of dollars to hours is 11/2 while the unit rate of hours to dollars is 2/11. Because both unit rates are valid, it is very important, in any particular context, to make a choice of the order of the variables and to be clear about which quantity is being the unit. To put it another way, using terminology from 6th grade mathematics: in any context, unless the context dictates which variable is independent, the solver must choose an

"independent variable" and stay with that choice throughout the discussion. In these terms the *unit rate* is the quotient r of the dependent variable (y) by the independent variable (x), and is described as r units of y per unit of x.

Example 6.

> Let's buy some eggs. A measure of the value of a certain quantity of eggs is its number: a dozen eggs has 12 times the value of a single egg, and a gross (twelve dozen) of eggs has 12 times the value of a dozen, or 144 times the value of a single egg. At his roadside stand, the farmer sells his eggs at \$3.84 a dozen. If you want to buy one egg, since it is $\frac{1}{12}$ of a dozen, it will cost you $\frac{1}{12} \times \$3.84 = \$0.32$. If you want to buy a gross of eggs, it will cost you $12 \times \$3.84 = \$46.08$.

In this example we use three different unit rates: cost per egg, cost per dozen and cost per gross, and we have illustrated how to change from one unit to the other. Depending upon the context, the unit measure of eggs could be "one egg" or "one dozen eggs" or "a gross of eggs." If I live alone, the cost per egg is most important to me but if I am a grocer, I want to know the cost per gross. Understanding and clearly communicating the unit rate and which unit it is taken with respect to are vitally important.

Example 7.

> Camila can ride a bike 5 miles in 20 minutes. Express her average speed as a unit rate of distance over time measured in minutes, and then fill in the missing entries in the table below.

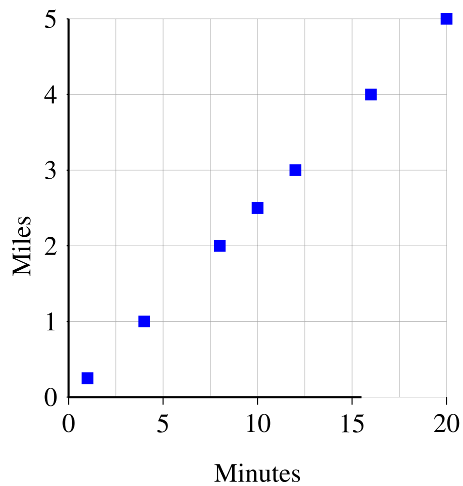
Minutes	20					10	1
Miles	5	1	2	3	4		

SOLUTION. The problem specifies "minutes" to be the independent variable. Since one minute is (1/20)th of twenty minutes, the distance to be associated to one minute is (1/20)th of 5 miles, or a fourth of a mile. So the unit rate is: $\frac{1}{4}$ mile per minute. We now can fill in the table, with the last unit showing unit rate in miles per minute.

Minutes	20	4	8	12	16	10	1
Miles	5	1	2	3	4	2.5	0.25

Again, notice the same patterns in the table as before. As the number of minutes increases by four (the unit rate of minutes per mile), the number of miles increases by one. When the number of minutes triples from four to twelve, the number of miles also triples from one to three. When the number of minutes is halved from 20 to 10, the number of miles is halved from five to two and one-half. The second numeric column shows the unit rate of 4 minutes per mile and the last column shows the unit rate of 0.25 miles per minute. Note the reciprocal relationship of the two unit rates.

When we graph these data, since the original problem was stated in terms of "miles per minute", we take "minutes" as the independent variable and "miles" as the dependent variable.



Miles per Minute

Another use of unit rates is in the analysis of real-world situations. Often we want to find the better deal in the grocery store by comparing two (or more) different sizes or different brands of a given item. Other times we want to determine which object is moving faster or which project is being completed more quickly. When two rates are given, it can be difficult to determine which rate is higher or lower because they have different values. It is not until both rates are converted to the same unit (a unit of one) that the comparison becomes easy. Once we have converted the rates to the same "language," the choice of which size or brand of product gives the best buy is easy. Similarly, once the speeds or rates of completion are measured in the same units, the result is readily apparent. This is illustrated in the next example.

EXAMPLE 8.

Camila's friend Xavier also likes to ride bikes. He can pedal 4 miles in 15 minutes. Who can ride faster, Camila or Xavier?

SOLUTION. At first glance, it is hard to tell who is quicker. From the given information, Xavier travels a shorter distance, 4 miles instead of 5 miles, but he does so in less time, 15 minutes instead of 20. We recall the computation of Camila's rate (of one mile in four minutes or one quarter miler per minute, that is 0.25 miles/minute). The computation for Xavier is

$$\frac{4 \ miles}{15 \ minutes} = \left(\frac{4}{15}\right)\left(\frac{miles}{minute}\right) = 0.267\frac{miles}{minute} \ ,$$

so Xavier rides a little faster than Camila (since he covers 0.267 miles in the same time that it takes her to ride 0.25 miles).

Another way to see this is to calculate the time each takes to ride one mile. it takes Camila 4 minutes to ride one mile where it takes Xavier 3.75 minutes to ride one mile. Since it takes her longer, she is slower.

As noted earlier, if we have equivalent ratios, we can use the unit rate to solve for desired quantities. We saw this in the examples involving cookies, yardwork, and cycling. We now introduce the concept of a

proportional relationship and take advantage of the unit rate to quickly compute the desired quantity.

> Two quantities x and y are in a *proportional relationship* if the quotient y/x is a fixed number r whenever x is not zero. This may also be written $y = rx$ or $x = y/r$ (when r is nonzero). In a proportional relationship, r is the *unit rate* of y with respect to x. This same unit rate, r, is also called the **constant of proportionality** (sometimes referred to as the proportional constant).

EXAMPLE 9.

Camila's club is sponsoring a "bike-marathon" where cyclists ride for 26.2 miles. Given Camila's average rate of 5 miles in 20 minutes, how much time will it take her to complete the race?

SOLUTION. As noted above, there are two possible unit rates: miles per minute and minutes per mile. Assuming she will be able to cycle at this rate for the entire race, the ratio of time to miles will be a constant, 4 minutes per mile. Letting y be the time, x be the distance and using the constant of proportionality $r = 4$, we quickly compute that $y = 4\frac{min}{mile} \cdot 26.2mile = 104.8$ minutes or one hour and 44.8 minutes.

Note that the argument says that when we scale up 1 mile to 26.2 miles, the time it takes (4 minutes) scales up to $26.2 \cdot 4$ minutes; in other words, for Camila, "miles cycled" and "time it takes" are both measures of the same distance so are proportional, with unit rate one-fourth mile per minute.

To summarize the content of this section, two variables y and x are said to be *associated,* if the values of these variables are paired. To say that they are in the *ratio* $a : b$ is to say that for every pair of associated values, the fraction y/x is equivalent to the fraction a/b. The *unit rate* of y with respect to x is $r = a/b$. If we shift our interest from specific pairs of values to the behavior of the variables with respect to each other, we shift the language to that of *proportion*. We say that the variables y and x are *proportional* if the quotient y/x is constant for all pairs of associated values. That constant is called the

constant of proportionality. Thus ratio and proportion describe the relation between two variables that are associated, but put focus on different aspects of that association.

Extension

In Example 9, we asked for the change from one measure of a race to another: from distance traveled to time expended, assuming a constant rate of speed (in that case, one-fourth of a mile per minute. This kind of problem is central to the physical sciences, and leads to what is called *dimensional analysis*; that is the analysis of a situation by means of various measures. This subject develops ways of keeping track of one set of units to another. To illustrate:

Example 10.

There are three feet in a yard. So, if something measures 1 yard, it is 3 feet long. If something else measures 4.5 yards, it is $4.5 \times 3 = 13.5$ feet long. In general, if something measures x yards, it measures $3x$ feet. A way to remember this is:

$$\frac{\text{Feet}}{\text{Yards}} = 3 ,$$

and for any object being measured,

$$\text{Feet} = (\frac{\text{Feet}}{\text{Yards}}) \cdot \text{Yards} .$$

In short, if we change from yards to feet, we multiply by the number of feet in a yard; that is, we multiply by the *unit rate* of feet to yards. In Example 9, we illlustrated a central fact: that (at constant speed), the unit rate of miles per minute is the inverse of the unit rate of minutes per mile: the rate of 4 minutes per mile is the same as (1/4) mile per minute. This is of course, true in general: if quantities X and Y are proportional, then the unit rate of X with respect to Y is the inverse of the unit rate of Y with respect to X. That leads to a process of cancellation, when making a major change in measures, that is the basis of dimensional anaysis. Let's illustrate this through some examples.

Example 11.

There are 5280 feet in a mile. How many yards in a mile?

Solution.

$$1 \text{ mile} = 5280 \text{ feet} = 5280 \text{ feet} \times \frac{1 \text{ yard}}{3 \text{ feet}} = \frac{5280}{3} \text{ yards} = 1760 \text{ yards} .$$

The move from the third expression to the fourth is to be thought of as simply cancellation of the measure "feet."

Example 12.

There are 12 inches in a foot, and there are 39.37 inches in a meter. How many yards are in a meter?

Solution. We have been given these data:

$$\frac{\text{Inches}}{\text{Feet}} = 12 , \quad \frac{\text{Inches}}{\text{Meter}} = 39.37 \quad \text{and we know} \quad \frac{\text{Feet}}{\text{Yards}} = 3 .$$

We want to find Yards/Meter. We can make this change (since each unit rate is a scale factor):

$$\frac{\text{Yards}}{\text{Meters}} = \frac{\text{Yards}}{\text{Feet}} \cdot \frac{\text{Feet}}{\text{Inches}} \cdot \frac{\text{Inches}}{\text{Meters}} = \frac{1}{3} \cdot \frac{1}{12} \cdot 39.37 = 1.09 .$$

Two variables x and y are in a proportional relationship if y/x is a constant. This constant is referred to as "units in y per units in x," or, the *unit rate* of y to x. To change from a value of x to a value of y, multiply by the constant.

Example 13.

Convert miles/hour into feet/second. We know

$$\frac{\text{Feet}}{\text{Miles}} = 5280\,, \quad \frac{\text{Seconds}}{\text{Hours}} = 3600\,,$$

so

$$\frac{\text{Feet}}{\text{Seconds}} = \frac{\text{Feet}}{\text{Miles}} \cdot \frac{\text{Miles}}{\text{Hours}} \cdot \frac{\text{Hours}}{\text{Seconds}} = 5280 \cdot \frac{\text{Miles}}{\text{Hours}} \cdot \frac{1}{3600} = \frac{5280}{3600} \cdot \frac{\text{Miles}}{\text{Hours}} = 1.467 \cdot \frac{\text{Miles}}{\text{Hours}}\,.$$

It is easier to remember that 5280/3600 = 88/60. (That is 60 miles per hour is the same as 88 feet/second).

End Extension

Section 4.2: Construct and Analyze the Representations of Proportional Relationships

Recognize and represent proportional relationships between quantities. 7.RP.2.

- *a. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.*
- *b. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.*
- *c. Represent proportional relationships by equations.*
- *d. Explain what a point (x, y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points $(0, 0)$ and $(1, r)$ where r is the unit rate.*

In this section, students build on proportional relationships as developed in section 4.1 by exploring, in contexts, representations of the relationship using tables and graphs. From these tables and graphs, students determine whether a context represents a proportional relationship and begin to create an equation to model proportional situations (further developed in section 4.3). The goal of this section is to have students move fluidly between the table, graph, equation (if proportional) and context.

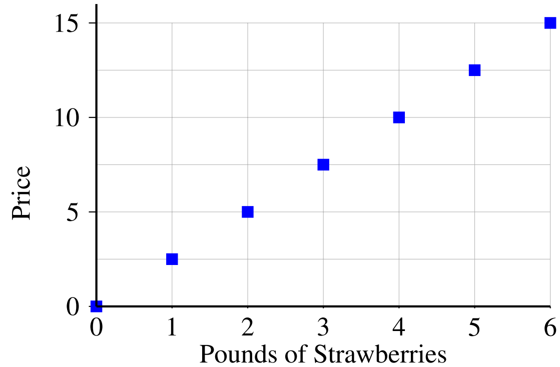
Example 14.

Jonathan loves strawberries. At the store, strawberries sell for \$2.50 per pound. Create a table and then draw a graph for this situation.

Solution.

Pounds of strawberries	0	1	2	3	4	5	6
Price (\$)	0	2.50	5.00	7.50	10.00	12.50	15.00

Using the data for this problem, we calculate that the ratio of price to pounds of strawberries in each nonzero column is (5/2):1. Because of this constant unit rate, the graph consists of points on a line that goes through the origin, (0, 0). This represents the fact that, if Jonathan doesn't buy any strawberries, he doesn't have to pay anything. Also, note that the unit rate of price to pounds of strawberries is represented at the point (1, 5/2). These two traits, the graph is a line going through the origin and the unit rate at the point (1, *r*), are characteristics of every proportional relationship.



Cost of Strawberries

Note that, when graphing data given in a table, the way the table is set up determines which quantity goes on the horizontal axis and which goes on the vertical. The convention is that the first quantity (on top for a horizontal table and on the left for a vertical table) goes on the horizontal axis. When a problem is given in a context, the solver gets to choose which quantity goes on which axis according to the sense of the context.

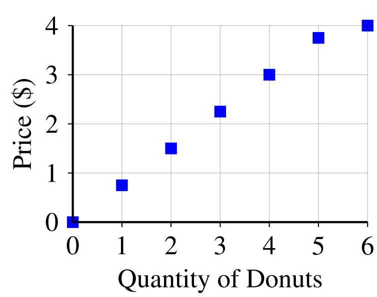
EXAMPLE 15.

Tasty-Yums sells assorted donuts according to the prices in the table. Determine if price and quantity of donuts are proportional.

Quantity	0	1	2	3	4	5	6
Price ($)	0	.75	1.50	2.25	3.00	3.75	4.00

SOLUTION. Computing the unit rates of price per quantity for 1 to 5 donuts yields \$0.75 per donut. However, the rate for 6 donuts is \$4.00 per half dozen which comes to a per-donut rate of \$0.66. So, the company gives a discount for buying half a dozen donuts, and may advertise this by writing "Donuts: \$0.75 each, or \$4 for a half dozen."

Here is a graph of the data in the above table:



Cost of Tasty-Yums Donuts

Since the last point $(6, 4.00)$ does not fit on the line segment emanating from $(0, 0)$, this is not a proportional relationship once we purchase 6.

The previous two examples show that some relationships are proportional and others are not. Often a proportional relation is not stated, but is implied by giving a few illustrations that show a pattern. Students benefit from determining whether a relationship is proportional by asking: "Is there a constant unit rate?" or "Do the points all lie on a line that goes through (0, 0)?" If the relationship is proportional, we can take the next step and begin to write an equation to express the relationship. This is illustrated in the next example.

EXAMPLE 16.

Amber is studying the relationship between the distance she covers and the number of steps to span that distance. She assumes that “steps taken” is a reliable measure of distance; that is “steps taken” will be proportional to any measure of the distance covered. This amounts to assuming that each of her steps covers the same distance. The table below shows the information she has gathered from several little walks.

# steps taken	0	10	20	30	40	1		x
# feet walked	0	16	32	48	64		1	

a. Determine the pattern and fill in the rest of the table.

b. Calculate the unit rates from the data.

c. Write an equation showing how one variable is related to the other.

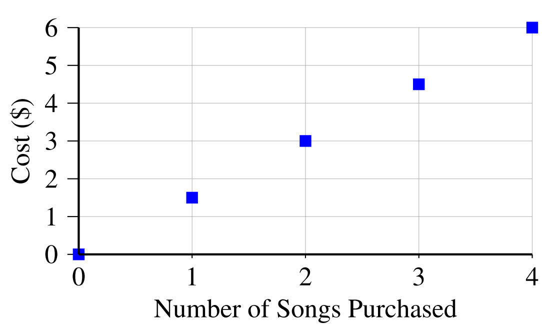
SOLUTION. It is fair to assume that Amber’s stride, the distance in feet covered by each step, is constant. Stride, measured in feet per step, is the unit rate we seek. If we take the first (nonzero) data point, we find that feet per step is 1.6, and we note that each given data pair confirms this; our confidence that Amber’s stride is constant is rewarded. Notice that the unit rate of 1.6 feet per step is precisely the multiplicative factor to go from the top row of the table to the corresponding cell in the bottom row. Therefore, if x steps are taken, Amber will have walked $1.6x$ feet. This information is included in the table. Note that we are simply scaling up or down the unit rate of 1.6 feet per step taken.

# steps taken	0	10	20	30	40	1	0.625	x
# feet walked	0	16	32	48	64	1.6	1	$1.6x$

Based on the table, we have found the equation relating number of feet walked to number of steps taken. Letting y be the number of feet walked (second row) and x be the number of steps taken (first row), we see (last data point) that $y = 1.6x$. Note that the unit rate is the coefficient of x (in symbols, $r = y/x$). This will always be the case when the two variables are proportional.

As noted in the introduction, the study of proportional relationships lays a foundation for the study of functions, to begin in Grade 8 and continuing through high school and beyond.

Linear functions are characterized by having a constant rate of change (the change in the outputs is a constant multiple of the change in the corresponding inputs). Proportional relationships are a major type of linear function; they are those linear functions that have a positive rate of change and take 0 to 0.



Cost of Purchasing J-Tunes Songs

Example 17.

The graph above shows the cost of the purchase of songs from the online store J-Tunes.

a. Are the data proportional?

b. Fill in the table below.

# Songs						
Cost ($)						

c. Find the unit rate of cost per song.

d. Create an equation for the cost of purchasing x songs.

Solution.

a. All the data points are on a line emanating from (0, 0) so the number of songs purchased and cost are proportional.

b. Reading from the graph gives the first five entries of the table below.

# Songs	0	1	2	3	4	n
Cost ($)	0	1.5	3	4.5	6	$1.5n$

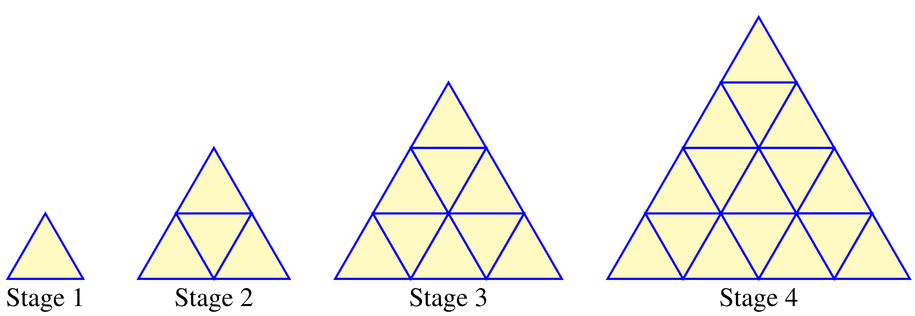
c. The second entry in the table tells us that one song costs $1.50, so the unit rate is 1.5 dollars per song. n songs will cost 1.50n$ dollars, as shown in the last entry.

d. Letting C be the cost of the purchase of n songs, we can write this as an equation: $C = 1.5n$.

Now we move to the analysis of scenarios using knowledge about proportional relationships. Geometric configurations provide a good place to explore patterns and reinforce concepts. In the next example, we explore a pattern, look for structure and reinforce the understanding of "perimeter."

Example 18.

Jennifer's young cousin is playing with triangular blocks. She makes the following figures and asks Jennifer if there are any patterns.



a. Fill in the table relating stage to perimeter and number of triangular pieces.

Stage	1	2	3	4	5	6	7	8
Perimeter	3	6						
# pieces	1	4						

b. Create graphs relating stage to perimeter and stage to number of triangular pieces.

c. Make an educated guess (using the above figure) that relates perimeter to the stage number.

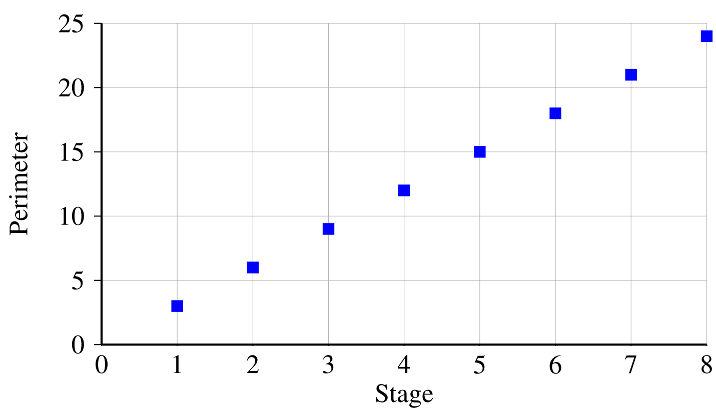
d. Are there any proportional relations in these data? Explain why or why not.

SOLUTION. Notice that units have not been specified, because we are interested in how the figures are numerically related, and whether the measure of the sides is in feet, yards or kilometers is not of interest. So we implicitly take, as the unit of length, the length of a side of the triangle in Stage 1.

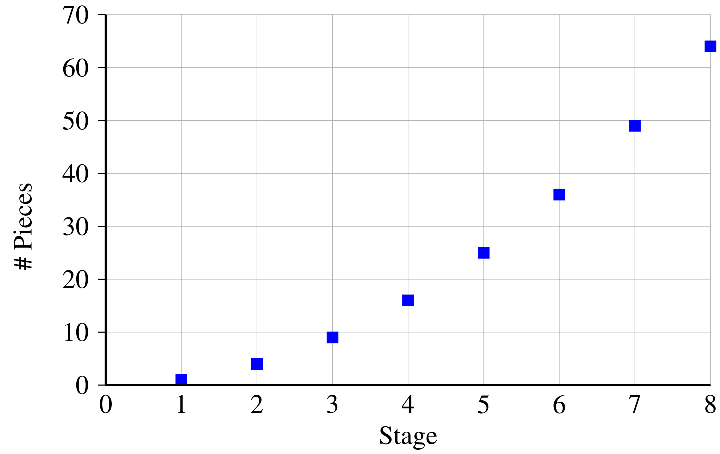
a. The first four data points are found by counting. We can continue by repeating the pattern four more times, or by noting that we move from one stage to the next by drawing a new row of triangles at the bottom. One way of describing the transition from one picture to the next is that it consists of adding a new (bottom) row. The new row consists of flips of all the triangles in the previous bottom row plus two new triangles at the ends. So, the number of triangles on the new bottom row is the next odd number. That tells us how to fill in the bottom row: calculate each entry, by adding the next odd number to the entry just calculated. As for the perimeter, we notice that the move from one stage to the next adds one Stage 1 side length to each side. Since there are three sides to the large triangle, the perimeter increases (from one stage to the next) by 3 Stage 1 side lengths. Thus we finish the second row of the table by adding 3 as we move from Stage to Stage.

Stage	1	2	3	4	5	6	7	8
Perimeter	3	6	9	12	15	18	21	24
# pieces	1	4	9	16	25	36	49	64

b. Graph



Stage and Perimeter



Stage and # Pieces

c. As we look at the table, for every column the entry in the second row (perimeter) is 3 times the stage number. That tells us that $P = 3S$, and so P and S are in the same rato, 3:1. As for the number of pieces per stage, the ratio changes from column to column, so the number of pieces is *not* proportional. We should recognize the pattern of the squares of the stage number.

d. The graphs of the data indicate to us that perimeter is proportional to stage number (with ratio $3:1$) while number of pieces is not proportional to stage number.

One of the practical uses of finding equations is the comparison of data. Real world examples include comparing different financial options. Below is such an example: comparing cell phone plans. Note that the graphs of the two options are both straight lines, but only one is the graph of a proportional relation.

Example 19.

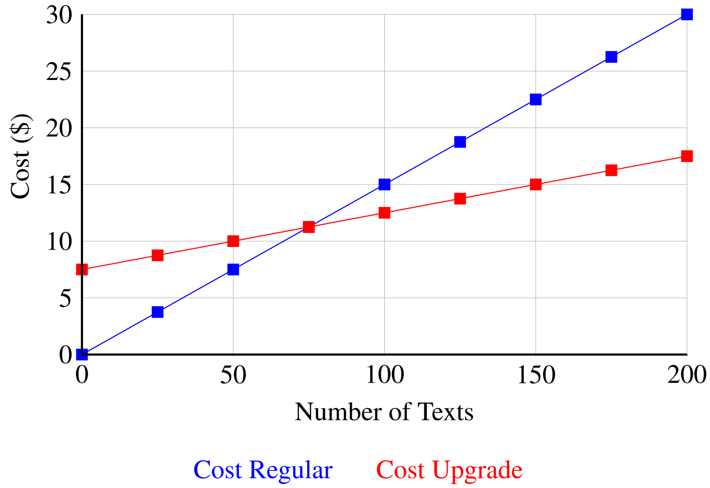
Grace is researching texting options for her cell phone. With her carrier she pays \$0.15 per text on the basic plan. She can upgrade to a texting plan by paying a flat \$7.50 each month and just 5 cents for each text sent. Create a table to analyze the costs of texting on each plan. Which plan should Grace sign up for? Your answer will reference how many texts she plans to send/receive each month.

Solution. The graph that shows these data together is this; looking at the graph we note that Grace is spending less on the regular plan so long as she sends no more than 75 texts a month. If she sends more than that, the regular plan is more costly.

Section 4.3: Analyze and Use Proportional Relationships and Models to Solve Real-World Problems

Use proportional relationships to solve multistep ratio and percent problems. Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error. 7.RP.3

The concepts studied in the previous sections will be applied throughout this section. Students will set up and solve proportions for real-world problems, including problems with percentages of increase and decrease. It's likely, especially for advanced classes, that before the topic is formally introduced students will set up proportions and solve them (a) using properties of equality, (b) by finding a common denominator on both sides, or even (c) by cross multiplying, if they have seen this in previous years or at home. Any method that can be justified using



proportional reasoning can lead to a meaningful discussion. Students should be encouraged to justify their answer with other representations (graph, bar model, table, or unit rates) .

EXAMPLE 20.

During the summer, Jordan helps a neighbor with yard work. One day Jordan worked for three hours and earned \$16.50. At this rate, how long will Jordan need to work to have enough money to buy an MP3 player that costs \$88? Set up a proportion equation for this situation and solve.

SOLUTION. First we calculate, using the given data, the unit rate of money earned per hour worked:

$$\text{unit rate} = \frac{\$16.50}{3 \textit{ hours}} = \$5.50 \text{ per hour.}$$

So, if Jordan works two hours, he earns $2 \times \$5.50 = \$11$; if he works 6 hours, he earns $6 \times \$5.50$, and if he works x hours, he earns $x \times \$5.50$. Since he wants to earn \$88, we should solve $5.50x = 88$, giving $x = 16$ hours.

EXAMPLE 21.

In a classroom of twenty-seven students, the ratio of those who do not own a cell phone to those who do is 1 : 2. How many students in this class own a cell phone?

SOLUTION. First note that the ratio 1 : 2 is that of the number of students without cellphones to the number of students with cellphones. Let x denote the number of students without cellphones. Then there are $2x$ students with cellphones and a total of $x + 2x$ students. Since that number is 27, we must solve $x + 2x = 27$, giving $x = 9$. We conclude that there are 9 students without cellphones and 18 with cellphones.

Ratios often occur in this way to describe the proportion of people in a given population that have a certain attribute as opposed to those who do not. For example, the ratio of right-handed people to left-handed people is 9 : 1. This is called a *part-to-part* ratio, in that it compares the sizes of different parts of a given population. If instead we said, “one in ten people is left-handed,” we’d be saying the same thing, but as a *part-to-whole ratio*. In both cases, we are saying that 90% of the population is right-handed. Be careful in working with ratios expressed in this way. The statements, “there are 9 times as many right-handed people as left-handed” and “10% of the population is left-handed” seem not to say the same thing, but they do.

Example 22.

The weather data for July in Provo, Utah show that for every 7 mostly clear days there are 3 overcast days. This datum is expressed as a part-to-part ratio. To rephrase as a part-to-whole ratio, we might say that we can have a picnic in Provo in July, 7 out of 10 days. Another way: typically in Provo, July will have about 9 overcast days.

Proportional relationships come up when considering tax and commission rates, markups and discounts. In all of these cases, the information is given as a unit rate expressed as a percentage. That means that the "unit" is taken to be 100; a tax rate of 6.75 percent just says that the tax due on \$100 is \$6.75. We have seen a set of problems about percentages in chapters 1 and 3, here we point out that in those problems we are discussing proportional relationships without mentioning proportion. Let us now look at some of these examples of proportional reasoning.

Example 23.

Markup in the retail industry means the percentage of cost for an item that is added to set the sale price of the item. To illustrate, suppose the markup is 12 %. That tells us that for every \$100 spent on acquiring goods, the manager adds on \$12 to create the store's profit. That is the same as saying that the markup is "12 cents on the dollar" – meaning that for an item that costs the store, say \$88, the markup will be $0.12 \times 88 = 10.56$ dollars.

In general, if the markup is $m\%$, then the sale price for an item costing C dollars will be

$$C + \frac{m}{100}C$$

Example 24.

A sporting goods store is advertising a sale of 30% off every regularly priced item. Kelly wants to purchase some new apparel and equipment.

a. Use a proportion to solve for the sale price of a shirt that is normally priced at \$19.50.

b. Complete the following table using your choice of solution method.

Original Price (x)	\$19.50	\$25.00	\$43.20	\$9.90	\$8.48	\$29.99
Sale Price (y)						

c. What is the unit rate of (sale price)/(original price)?

d. Write an equation for y, the sale price, for an item that has a regular price of x.

e. If the sale price for running shoes is \$48.23 what was the original price?

Solution.

a. If the regular price is \$19.50, then the discount, d, (the part) can be found using the percent proportion equation

$$\frac{30}{100} = \frac{d}{\$19.50} \quad \text{so} \quad d = 0.30 \cdot 19.50\,.$$

Solving gives $d = \$5.85$. Since the discount is \$5.85, the sale price must be \$19.50 − \$5.85 or \$13.65. Alternatively, one might look at the sale price, s, as the part. Since there is a 30% discount, Kelly will pay 100% − 30% = 70% *of the original price*. The percent proportion equation would be

$$\frac{70}{100} = \frac{s}{\$19.50}$$

Solving this proportion yields $s = \$13.65$, the same result.

b.

Original Price (x)	\$19.50	\$25.00	\$43.20	\$9.90	\$8.48	\$29.99
Sale Price (y)	\$13.65	\$17.50	\$30.24	\$6.93	\$5.93	\$20.99

c. Since we know this is a proportional relationship, it suffices to calculate the ratio for one instance, the unit rate is $13.65/19.50 = .70$, or 70%. Wait a minute – was there an easier way to compute this?

d. Letting S represent the sale price and R the regular price, $S = 0.7P$.

e. In this part of the problem we are given the sale price, S of \$48.23, and are looking for the original price P. So, we have to solve $48.23 = 0.7P$. One way to figure this out is to rearrange the previous equation using the multiplicative inverse of 0.7 to get $P = \frac{10}{7}S$. Then we substitute in 48.23 for S and get $P = 68.90$. We can check our answer by multiplying 68.90 by 0.7 to get 48.23. Therefore we have verified that the original price was \$68.90.

In general, if the discount on an item is $d\%$, then the sale price for an item originally priced at S dollars will be

$$S - \frac{d}{100}S$$

Example 25.

If the population today of Isla Incognita is 113% of what it was in the year 2000 and the population in 2000 was 36,000, then today's population is $1.13 \times 36,000 = 40,680$. Suppose I know the present population of Isla Linda to be 18,000, and I also know that it is 113% of the population in 2000. What was the population in 2000?

Solution. We have to solve the equation $1.13 \times P = 18,000$, or $P = 18,000/1.13 = 15,929$. Be careful! You might have argued, "if the population grew by 13% from 2000 to today, that means that the 2000 population was 13% less then," so you calculate $(1 - 0.13) \times 18,000 = 15,660$, a quite different figure, and the wrong one.

This possible mistake is best illustrated by a simple example. If there are three people in a room, and another person enters the room, the population increases by 33%, for 4 is $4/3 = 1 + 1/3$ of 3. But now, if that person leaves the room, the population decreases by 25% because $3 = 3/4$ of 4, or $1 - 1/4$ of 4.

Example 26.

The population of the US is 313.9 million, and that of Brazil is 190.7 million. Express in percentages how much larger the US is than Brazil and how much smaller Brazil is than the US.

Solution. Choose p so that the US is $p\%$ larger than Brazil. Then we have to solve

$$\left(1 + \frac{p}{100}\right)(190.7) = 313.9$$

Now let q be such that Brazil is $q\%$ smaller than the US. Here we have to solve

$$\left(1 - \frac{q}{100}\right)(313.9) = 190.7$$

The answers are $p = 64.6$ and $q = 39.2$.

Example 27.

Two stores have the same skateboard on sale. The original price of the skateboard is \$200. At store AAA, it is on sale for 20% off with a rewards coupon that allows the purchaser to take an additional 30% off the sale price at the time of purchase. At store BBB, the skateboard is on sale for 30% off. They too have a rewards coupon, but their coupon is for an additional 20% off the sale price. Will the price for the skateboard be the same at both stores? If not, which store has the better deal?

Solution. The problem presents two scenarios, at stores AAA and BBB, and asks us to compare the final price in each scenario. Let us go through each scenario in detail.

- At AAA, the original price is \$200. It is marked down 20%, which is $0.20 \times 200 = 40$ dollars, so is now priced at \$160. The purchaser with coupon can take off an additional 30%, which is $0.30 \times 160 = 48$ dollars. Thus the final price to the purchaser with coupon is \$112.
- The scenario at BBB is the same, but with the percentages interchanged. So, at BBB the first markdown is $0.30 \times 200 = 60$ dollars, and the sale price is \$140. The purchaser with coupon gets a further 20% discount, which is $0.20 \times 140 = 28$ dollars. Thus the final price is \$112.
- The result is that the deals are the same. There is a way to see that more easily. At AAA, the sale price is 0.80×200, and with the coupon, the final price is thus $0.70 \times 0.80 \times 200$ dollars. At BBB, the percentages are reverses, so the final price is $0.80 \times 0.70 \times 200$. But these numbers are the same because of the commutativity of multiplication.